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I.—ON THE GENERAL PROBLEM OF INTERCALATIONS.

1. It having been so ordered, for beneficent ends, as that no one of the three most obvious measures of time is a multiple of another, the necessity of astronomical intercalations was discovered very early; and various methods have been used to provide against the equinoxes and solstices wandering inconveniently far from assigned points in the civil year. So long as this is all that is proposed, the problem of intercalations is plainly indeterminate, and we may choose such a solution as appears to furnish the most simple rule for determining the length of any proposed year. The Gregorian intercalation answers this purpose very well; but it does not* fulfil the condition, evidently possible, that the sun shall be in the vernal equinox, or any other particular point of his orbit, during the same assigned space of twenty-four hours in each year. It does not distribute the leap-years as evenly as possible; insomuch that if we select from a period of 400 years that part which contains the greatest proportion of leap-years, namely that from A.D. $(4m-1)\,100+3$ to $400m+96$, we find that the equinoxes and solstices happen 2 days and nearly 6 hours earlier in the latter year than in the former. It is true that this is of no great practical consequence, but still a mathematician may well like to know *what is the most perfect system of intercalation*. The problem is one *sui generis*, and its solution is given in a rather intricate, but at the same time elegant, law. It may be enunciated generally as follows.

* It is surprising that so eminent an astronomer as Sir John Herschel should have asserted the contrary. See Dr. Lardner's Cabinet Cyclopædia, Astronomy, p. 410.

2. Given two numbers whose common measure is not greater than unity, it is required to determine a series of multiples of the smaller number, whose excesses above the successive terms of an arithmetic progression whose common difference is the greater number, shall be the least possible.

3. Let a and b be the two numbers, of which a is the greater; and let x be any integer. We have to find an integer y, such that $by-(ax-f0)$ may be the least possible positive quantity, that is, that it may lie between 0 and b; $f0$ being arbitrary, but contained between the same limits.

4. We may assume

$$by - ax + f0 = fx\,; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

then $fx > 0$, but $< b$. Taking the difference with respect to x,

$$b\Delta y - a = \Delta fx\,;$$

$$\text{therefore} \quad \Delta y = \frac{a+\Delta fx}{b}.$$

Now Δfx may be either positive or negative, but it must be arithmetically less than b. Let n be the whole number next less than $\frac{a}{b}$, then the preceding equation gives

$$\Delta y = n \text{ or } n+1, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

$$\text{and} \quad \Delta fx = nb - a \text{ or } (n+1)b - a,$$

$$= -(a-nb) \text{ or } b-(a-nb) \quad \ldots\ldots \quad (3).$$

5. Since Δfx is sometimes negative and sometimes positive, and fx always lies between 0 and b, its values must range continually between maxima and minima. Also, since one of the quantities $a-nb$, $b-(a-nb)$, must be greater than $\frac{1}{2}b$, that value of Δfx which, abstracted from its sign, is the greater, and the corresponding value of Δy, cannot exist for two successive values of x, or fx would transgress one of its limits. Hence, according as

$$a - nb > \text{ or } < b-(a-nb),$$

the minimum values of fx must immediately succeed the maximum, or conversely. Let z, $z+p$, be two successive values of x which render fx a minimum or maximum, accordingly; and let b' represent the least of the quantities $a-nb$, $b-(a-nb)$. Then while x increases from z to $z+p-1$, $\Delta fx = \pm b'$; and for the next term, $\Delta fx = \mp(b-b')$. Hence

$$f(z+p) = fz \pm (p-1)b' \mp (b-b'),$$

$$\text{or} \quad f(z+p) - fz = \pm pb' \mp b \quad \ldots\ldots\ldots \quad (4),$$

therefore $p = \frac{b}{b'} \pm \frac{f(z+p) - fz}{b'}$ ....(5).

6. In order to determine p, it is requisite to find the limits of the maximum and minimum values of fx, which is done as follows. If fz be a minimum,

$$f(z-1) = fz + b - b',$$

and this must be $< b$, therefore

$$fz < b' \text{ .... } (6);$$

again, if fz be a maximum,

$$f(z-1) = fz - (b - b'),$$

and this must be > 0, therefore

$$fz > b - b' \text{ .... } (7).$$

7. Thus, when fz and $f(z+p)$ are minima, they are each > 0 and $< b'$; and when they are maxima, they are each $> b - b'$ and $< b$. Therefore, in either case,

$$f(z+p) - fz > -b' \text{ and } < +b'.$$

Applying these limits to equation (5), it follows that

$$p > \frac{b}{b'} - 1, \text{ and } < \frac{b}{b'} + 1.$$

Let n' be the integer next less than $\frac{b}{b'}$. Then

$$p = n' \text{ or } n' + 1 \text{ .... } (8).$$

8. It remains to find when p has the one, and when the other, of these values. For this purpose, let x' be the number of values of z in the interval from 0 to x, so that while x increases from z to $z+p$, x' increases to $x'+1$; and let $f'x'$ be assumed

$$= fz \text{ or } b - fz \text{ .... } (9),$$

according as fz is a minimum or maximum value of fx; that is, according as $a - nb >$ or $< b - (a - nb)$. Then

$$\Delta f'x' = \pm \{f(z+p) - fz\},$$

which, by (4), $= pb' - b$.

Substituting the values of p from (8), we find

$$\Delta f'x' = -(b - n'b') \text{ or } b' - (b - n'b'),$$

equations exactly similar to those marked (3). Also we have, in either case, by (6), (7), and (9),

$$f'x' > 0, \text{ and } < b'.$$

9. Hence it follows, by the same reasoning as was before used in § 5, that according as $b - n'b' >$ or $< b' - (b - n'b')$, the value n' or $n' + 1$ of p cannot continue for two successive values of x', or, consequently, of z; and in order to carry the investigation further, we have only to change a, b, x, fx, n, in the preceding formulæ, into $b, b', x', f'x', n'$, and $z, p, b', n', x', f'x'$, into $z', p', b'', n'', x'', f''x''$, and so on. Thus we find, that the maximum and minimum values of fx will also have their maximum and minimum values, which we may call maxima and minima of the second order; and that every maximum and minimum of the second order corresponds to a change in the value of p. In like manner, every maximum and minimum of the third order corresponds to a change in the value of p', which represents the number of maxima or minima of the first order between two successive maxima or minima of the second order. By this method, though we cannot assign the successive values of y, *ad infinitum,* we may readily do so for a vast number of terms; since, by repeating r times the operations indicated above, we obtain the law of a number of successive terms greater than $nn'n''\,.\,.n^{(r)}$, as will be more fully explained hereafter.

10. It is desirable to have an easy method of finding the numbers n, n', n'', &c. For this purpose let $\frac{a}{b}$ be developed in a continued fraction; then the first quotient will be n, but the others will not always be n', n'', &c.

$$\text{Assume therefore}\quad \frac{a}{b} = n + \frac{1}{n_1 +}\,\frac{1}{n_2 +}\,\frac{1}{n_3 +}\ldots\ldots$$

$$\text{then}\quad a - nb = \frac{b}{n_1 +}\,\frac{1}{n_2 +}\,\frac{1}{n_3 +}\ldots\ldots$$

and according as $a - nb >$ or $< b - (a - nb)$,

$$\text{or}\quad a - nb > \text{ or } < \tfrac{1}{2}b,$$

$$\text{we find}\quad \frac{1}{n_1 +}\,\frac{1}{n_2 +}\,\frac{1}{n_3 +}\ldots\ldots > \text{ or } < \tfrac{1}{2},$$

$$\text{and}\quad n_1 + \frac{1}{n_2 +}\,\frac{1}{n_3 +}\ldots\ldots < \text{ or } > 2,$$

$$\text{and therefore}\quad n_1 = \text{ or } > 1.$$

Hence, if $n_1 = 1$, we have, by § 5,

$$b' = b - (a - nb)$$

$$= b - \frac{b}{1 +}\,\frac{1}{n_2 +}\,\frac{1}{n_3 +}\ldots\ldots;$$

therefore $$\frac{b}{b'} = \frac{1}{1-}\,\frac{1}{1+}\,\frac{1}{n_2+}\,\frac{1}{n_3+}\cdots\cdot$$

Let $$n_2 + \frac{1}{n_3+}\cdots\cdot = N,$$

then $$\frac{b}{b'} = \frac{1}{1-}\,\frac{1}{1+}\,\frac{1}{N} = \frac{1}{1-}\,\frac{N}{N+1} = N+1,$$

$$= n_2 + 1 + \frac{1}{n_3+}\,\frac{1}{n_4+}\cdots\cdot;$$

therefore (§ 7) in this case $n' = n_2 + 1$.

But if n_1 be greater than unity,

$$b' = a - nb,$$

and $$\frac{b}{b'} = n_1 + \frac{1}{n_2+}\,\frac{1}{n_3+}\cdots\cdot;$$

therefore in this case $n' = n_1$.

11. Hence we have the following rule for determining the numbers n, n', n'', &c.

Write down in order the successive quotients of the development of $\frac{a}{b}$ in a continued fraction. Whenever any quotient, other than the first, is unity, join it by addition to the following quotient. The series thus produced consists of the numbers required.

12. The law of the values of y may now be explained in any particular case. Suppose that n_1, n_2, n_3, &c. are all greater than unity. We have first, by (2), two values of Δy, namely n and $n+1$. The value $n+1$ can exist for one term only at a time (§ 5), but the value n, by equation (8), continues for either $n'-1$ or n' terms. Hence we have

First, a period of n' terms,

Secondly, a period of $n'+1$ terms,

both of which are terminated by a value of Δy equal to $n+1$.

Let these be called the smaller and greater periods of the first order.

Again we have two periods of the second order; the smaller containing $n''-1$ smaller periods of the first order, and one greater period of the first order; the greater containing n'' smaller periods of the first order, and one greater period of the first order.

Similarly the smaller period of the third order will contain $n'''-1$ smaller periods of the second order, and one greater

period of the second order; and the greater period of the third order will contain n''' smaller periods of the second order, and one greater period of the second order.

But if $n_1 = 1$, the periods of the first order will be terminated by a value of Δy equal to n, that function having been equal to $n + 1$ throughout the rest of the period. And if $n^{(r)}$ result from the addition of two consecutive quotients, of which the former is equal to unity, the smaller period of the r^{th} order will contain $n^{(r)} - 1$ *greater* periods of the $(r-1)^{\text{th}}$ order, and one *smaller* period of that order; and the greater period of the r^{th} order will contain $n^{(r)}$ *greater* periods of the $(r-1)^{\text{th}}$ order, and one *smaller* period of that order.

13. To complete the investigation, it remains that, knowing the value of $f0$ in (1), we should find the distance of the first terms of the periods of the several orders from the beginning of the series of the values of y, that is, that we should find the lowest values of z, z', &c.

We have then, according as $a - nb >$ or $< b - (a - nb)$, (§. 5), that is, according as $n =$ or > 1 (§ 10),

$$\Delta fx = \pm\, b'$$

from $x = 0$ till fx becomes $> b - b'$, or $< b'$. See (6) and (7). Hence if z_0 represent the lowest value of z,

$$fz_0 = f0 \pm (z_0 - 1)\, b' \mp (b - b')$$
$$= f0 \pm z_0 b' \mp b \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\; (10.)$$

Therefore $$z_0 = \frac{b}{b'} \pm \frac{fz_0 - f0}{b'}.$$

When the upper sign is to be taken,

$$fz_0 > 0, \text{ but } < b';$$

therefore $$z_0 > \frac{b - f0}{b'}, \text{ but } < \frac{b - f0}{b'} + 1;$$

therefore $z_0 =$ the integer next greater than $\dfrac{b - f0}{b'}$.

When the lower sign is to be taken,

$$fz_0 > b - b', \text{ but } < b;$$

therefore $$z_0 > \frac{f0}{b'}, \text{ but } < \frac{f0}{b'} + 1;$$

therefore $z_0 =$ the integer next greater than $\dfrac{f0}{b'}$.

If, in the first case, $f0$ should be less than b'; or, in the second, it should be greater than $b - b'$, the above formulæ will give n' or $n' + 1$ for the value of z_0. But the proper value of z_0 will on these suppositions be 0.

Having found z_0, fz_0 is known from equation (10); and we have next to find the place where the first period of the second order begins, or the value of z'_0. This is done in the same manner; only putting an additional accent on all the letters. $f'0$ is found from fz_0 by equation (9).

In the same manner the places where the periods of the third and higher orders begin, may be found.

14. As an example, let us take the question referred to in the beginning of this paper, that of determining the law of the length of the civil year, so that the place of the equinox may never vary twenty-four hours.

The ratio of the mean tropical year to the mean solar day has been found to be 365·242264. Let this be converted into a continued fraction, and we have the quotients

$$365,\ 4,\ 7,\ 1,\ 4,\ 1,\ 5,\ 1,\ \&c.$$

Hence (§. 11),

$$365,\ 4,\ 7,\qquad 5,\qquad 6,\qquad \&c.$$

are the respective values of

$$n,\ n',\ n'',\qquad n''',\qquad n'''',\qquad \&c.$$

Following the principles explained in §. 12, we find for the two periods of the first order,

A period of 4 years, containing 3 of 365 days, and 1 of 366;

A period of 5 years, containing 4 of 365 days, and 1 of 366.

For the periods of the second order,

A period of 29 years, containing 6 periods of 4 years and 1 of 5.

A period of 33 years, containing 7 periods of 4 years and 1 of 5.

For the periods of the third order,

A period of 161 years, containing 4 periods of 33 years and 1 of 29;

A period of 194 years, containing 5 periods of 33 years and 1 of 29.

For the periods of the fourth order,

A period of 1131 years, containing 5 periods of 194 years and 1 of 161;

A period of 1325 years, containing 6 periods of 194 years and 1 of 161.

15. Our knowledge of the length of the tropical year is hardly exact enough to enable us to carry the series further, if it were of any use to do so. To accomplish the purpose

fully, it would be necessary to take into account the variation of the tropical year, which might probably be done without much difficulty. The system of intercalation given above agrees with that adopted by the Persians, as far as the period of 161 years. See the *History of Astronomy* in the Library of Useful Knowledge, chap. x.

16. In order to determine what place a given year has in any period, the limits between which an equinox is always to fall, must be assigned. If it be required that the vernal equinox shall always happen on the twenty-first of March, in a given longitude, it is to be observed that according to the original assumption, $f0$ represents the interval between the time of the sun passing the equinoctial point and the following midnight; so that this being known accurately in any given year, the beginnings of the several periods may be found by the method explained in §. 13. To this end we have

$$\begin{array}{ll}
a = & 365 \cdot 242264 \\
b = & 1 \cdot 000000 \\
b' = a - nb = & \cdot 242264 \\
b'' = b - n'b' = & \cdot 030944 \\
b''' = (n'' + 1)\, b'' - b' = & \cdot 005288 \\
b'''' = (n''' + 1)\, b''' - b'' = & \cdot 000784
\end{array}$$

The reader who is interested in this subject, may apply the preceding formulæ to the measurement of time by years and lunar months, or lunar months and days. Other problems, besides astronomical, may also be solved by means of them; for instance, that of representing, as correctly as possible, an oblique line by stitches on canvass; and that of building a wall, the top of which shall follow a given slope, with horizontal courses of brick.

S. S. G.

II.—NOTE ON THE THEORY OF THE SOLUTIONS OF CUBIC AND BIQUADRATIC EQUATIONS.

By JAMES COCKLE, B.A. Trinity College.

IN the concluding section of Hymers' *Theory of Equations*, (1st Edit.) an outline is given of the method by which Lagrange (in the *Berlin Memoirs* for 1770,) succeeded in showing that all the particular and apparently isolated solutions of equations then known, were capable of being referred to one

general principle. The great interest attaching to the investigation induces me to hope, that the following extension of it, to one or two later cases, may not be unacceptable to the readers of this periodical.

Resuming then the equation (2), p. 249, vol. II. of the Journal, and multiplying both sides of it by the denominator of the right-hand side,

$$\{(n\rho)^{\frac{1}{3}} - \rho\} x = \{(n\rho)^{\frac{1}{3}} + a\} z + b$$

where $\rho = 3z + a$. Now let x_1, x_2, x_3, be the three values of x, and $\sqrt[3]{(n\rho)}$, $\alpha\sqrt[3]{(n\rho)}$, $\alpha^2\sqrt[3]{(n\rho)}$, the corresponding values of $(n\rho)^{\frac{1}{3}}$, then we have the following equations:

$$\{\sqrt[3]{(n\rho)} - \rho\} x_1 = \{\sqrt[3]{(n\rho)} + a\} z + b$$
$$\{\alpha . \sqrt[3]{(n\rho)} - \rho\} x_2 = \{\alpha . \sqrt[3]{(n\rho)} + a\} z + b$$
$$\{\alpha^2 . \sqrt[3]{(n\rho)} - \rho\} x_3 = \{\alpha^2 . \sqrt[3]{(n\rho)} + a\} z + b.$$

Add these equations, then, since $x_1 + x_2 + x_3 = - a$, and $1 + \alpha + \alpha^2 = 0$, we have

$$\sqrt[3]{(n\rho)}(x_1 + \alpha x_2 + \alpha^2 x_3) + \rho a = 3az + 3b,$$

$\therefore$ substituting, transposing, and making $x_1 + \alpha x_2 + \alpha^2 x_3 = Y$;

$$\sqrt{\{(a^2 - 3b)(3z + a)\}}.\ Y = -(a^2 - 3b),$$

$$\therefore z = -\frac{a}{3} - \frac{(a^2 - 3b)^2}{3Y^3} \ldots\ldots\ldots\ldots (1).$$

Hence this solution, like others, is effected by forming a "reducing equation," whose roots are functions of Y^3, which quantity has (Hymers, pp. 189—192) only two values. If in the expression for x we write for z its value derived from (1), a simple and obvious reduction will give us the same values of x as are obtained at p. 192 of Hymers; and the fact of only one of the values of Y^3 entering into the expression for z, confirms my concluding remark in the last number of this publication.

On examining next the discussion of a biquadratic, (Hymers, p. 192), we see that there are three several systems of functions of the roots of the original equation, which, possessing only three values, are competent to form the roots of the "reducing equation;" these functions are of the forms

$$(x_1 + x_3)(x_2 + x_4),\quad x_1x_3 + x_2x_4,\quad \text{and}\quad (x_1 - x_2 + x_3 - x_4)^2,$$

which last is the square of y, and forms the roots of the reducing equation in Euler's method; the second applies to Waring's, and the first to the accompanying one, whose final cubic (provided the elimination be performed as below) is *essentially* different from the ordinary ones and coincides with

that in Hymers', p. 192, line 6 from the bottom; to which indeed it is the corresponding particular solution.

Besides the above, there is another more complicated function, having only three values, which forms the basis of the solution given by your correspondent ʃʃ; it is of the form $\frac{x_1x_3 - x_2x_4}{x_1 - x_2 + x_3 - x_4}$, as he has shown in vol. I. of the Journal.

Now, let $x^4 + px^3 + qx^2 + rx + s = 0$

be supposed to be made up of the factors

$$(x^2 + ex + f)(x^2 + gx + h);$$

then, multiplying and equating coefficients of like powers of x,

$$e + g = p \ldots\ldots (1), \quad f + h + eg = q \ldots\ldots (2),$$
$$fg + eh = r \ldots\ldots (3), \quad fh = s \ldots\ldots (4);$$

but, $e - g = \sqrt{\{(e+g)^2 - 4eg\}} = \sqrt{(p^2 - 4eg)}$ by (1),

$f - h = \sqrt{\{(f+h)^2 - 4fk\}} = \sqrt{\{(q - eg)^2 - 4s\}}$ by (2) and (4);

and since (3) may be put under the form

$$r = (f + h)\frac{e+g}{2} - (f - h)\frac{e-g}{2};$$

and by substitution and transposition,

$$\sqrt{(p^2 - 4eg)}\sqrt{\{(q - eg)^2 - 4s\}} = p(q - eg) - 2r;$$

therefore squaring, transposing, and making $eg = z$,

$$z^3 - 2qz^2 + (pr + q^2 - 4s)z + p^2s - pqr + r^2 = 0 \ldots\ldots (2).$$

Had we made $f + h = z$, we should have had Waring's solution, or if we had taken $e - g$, we should have had the resulting cubic given in pp. 166 and 170 of Hymers, which is the basis of Euler's method; the process of elimination in each of these cases being of course slightly different.

Middle Temple, December 23, 1841.

III.—EXPOSITION OF A GENERAL THEORY OF LINEAR TRANSFORMATIONS.—PART II.

By George Boole.

From the great length to which the investigations of Part I. have extended, I shall in this paper chiefly confine myself to the exhibition of results, and shall leave it to the reader to supply the demonstrations omitted.

1. Let us represent the binary system of equations, $q = r$, and $Q = R$, under the forms

$$\Sigma k a_1 x_1^{\alpha} x_2^{\beta} \ldots . x_m^{\mu} = \Sigma k b_1 y_1^{\alpha} y_2^{\beta} \ldots . y_m^{\mu} \ldots \ldots . (1),$$

$$\Sigma k A_1 x_1^{\alpha} x_2^{\beta} \ldots . x_m^{\mu} = \Sigma k B_1 y_1^{\alpha} y_2^{\beta} \ldots . y_m^{\mu} \ldots \ldots . (2),$$

wherein a_1, b_1, A_1, B_1, are independent constants, susceptible each of a different value for every successive term of the function to which it belongs, k a numerical constant determined by the formula

$$k = \frac{1.2.3 \ldots . n}{1.2 \ldots . \alpha . 1.2 \ldots . \beta \ldots . 1.2 \ldots . \mu},$$

and $\alpha, \beta, \ldots . \mu$, indeterminate positive integers, (the value 0 included,) subject to the condition of homogeneity,

$$\alpha + \beta \ldots . + \mu = n.$$

The relations among the constants of (1) and (2) derived from the dependent conditions, $\theta(Q + hq) = 0$, $\theta(R + hr) = 0$, may then be expressed under the symbolical forms,

$$\frac{\left(\Sigma a_1 \dfrac{d}{dA_1}\right)^{\eta} \theta(Q)}{\theta(Q)} = \frac{\left(\Sigma b_1 \dfrac{d}{dB_1}\right)^{\eta} \theta(R)}{\theta(R)} \ldots . (3);$$

the values of η varying from 1 to γ the index of the degree of $\theta(Q)$. To elucidate more fully this notation, the above theorems may be compared with their equivalents in Part I., it being observed that $\dfrac{\theta(Q)}{\theta(q)} = \dfrac{\theta(R)}{\theta(r)}$.

2. The above results may be included in a very important generalization. Suppose that we have $\rho + 1$ equations, expressed under the general forms

$$\left.\begin{array}{lll} \Sigma k a_1 x_1^{\alpha} x_2^{\beta} \ldots x_m^{\mu} = \Sigma k b_1 y_1^{\alpha} y_2^{\beta} \ldots y_m^{\mu} \ldots & \text{for} & q_1 = r_1 \\ \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots & & \\ \Sigma k a_\rho x_1^{\alpha} x_2^{\beta} \ldots x_m^{\mu} = \Sigma k b_\rho y_1^{\alpha} y_2^{\beta} \ldots y_m^{\mu} \ldots & & q_\rho = r_\rho \\ \Sigma k A x_1^{\alpha} x_2^{\beta} \ldots x_m^{\mu} = \Sigma k B y_1^{\alpha} y_2^{\beta} \ldots y_m^{\mu} \ldots & & Q = R \end{array}\right\} \ldots (4);$$

then are the equations

$$\theta(Q + h_1 q_1 + h_2 q_2 \ldots + h_\rho q_\rho) = 0. \quad \theta(R + h_1 r_1 + h_2 r_2 \ldots + h_\rho r_\rho) = 0 \ldots (5),$$

identical relatively to h_1, h_2, … h_ρ; and if we put

$$\Sigma\left(a_1 \frac{d}{dA}\right) = \phi_1, \quad \Sigma\left(b_1 \frac{d}{dB}\right) = \psi_1,$$

$$\Sigma\left(a_\rho \frac{d}{dA}\right) = \phi_\rho, \quad \Sigma\left(b_\rho \frac{d}{dB}\right) = \psi_\rho;$$

then are all the relations among the constants of (4) included in the theorem

$$\frac{\phi_1^a \phi_2^b \ldots\ldots \phi_\rho^r\, \theta(Q)}{\theta(Q)} = \frac{\psi_1^a \psi_2^b \ldots\ldots \psi_\rho^r\, \theta(R)}{\theta(R)} \quad \ldots\ldots (6),$$

$a, b, \ldots\ldots r$, being indeterminate positive integers, the value 0 included, subject only to the condition that their sum shall not exceed γ. In place of (6), we may with somewhat greater generality say,

$$\frac{F(\phi_1 \phi_2 \ldots\ldots \phi_\rho)\, \theta(Q)}{\theta(Q)} = \frac{F(\psi_1 \psi_2 \ldots\ldots \psi_\rho)\, \theta(R)}{\theta(R)} \quad \ldots\ldots (7);$$

F denoting any rational and interpretable combination of the symbols to which it is affixed.

(1) Let us take the ternary system of equations,

$$\left.\begin{aligned} ax^2 + by^2 + cz^2 + 2dyz + 2exz + 2fxy &= a'x'^2 + \&c. \\ a_1x^2 + b_1y^2 + c_1z^2 + 2d_1yz + 2e_1xz + 2f_1xy &= a'_1x'^2 + \&c. \\ a_2x^2 + b_2y^2 + c_2z^2 + 2d_2yz + 2e_2xz + 2f_2xy &= a'_2x'^2 + \&c. \end{aligned}\right\} \ldots\ldots (8);$$

the first of which, $q = r$, we shall put in place of $Q = R$, of (4), then

$$\phi_1 = a_1 \frac{d}{da} + b_1 \frac{d}{db} + \ldots\ldots + f_1 \frac{d}{df} \qquad \psi_1 = a'_1 \frac{d}{da'} + \&c.$$

$$\phi_2 = a_2 \frac{d}{da} + b_2 \frac{d}{db} + \ldots\ldots + f_2 \frac{d}{df} \qquad \psi_2 = a'_2 \frac{d}{da'} + \&c.$$

$$\theta(q) = abc + 2def - (ad^2 + be^2 + cf^2) \qquad \theta(r) = a'b'c' + \&c.$$

As $\theta(q)$, $\theta(r)$ are of the third degree, it is manifest that all the forms deducible from (6) will be included in the following nine equations,

$$\frac{\phi_1\theta(q)}{\theta(q)} = \frac{\psi_1\theta(r)}{\theta(r)}, \qquad \frac{\phi_2\theta(q)}{\theta(q)} = \frac{\psi_2\theta(r)}{\theta(r)},$$

$$\frac{\phi_1^2\theta(q)}{\theta(q)} = \frac{\psi_1^2\theta(r)}{\theta(r)}, \qquad \frac{\phi_2^2\theta(q)}{\theta(q)} = \frac{\psi_2^2\theta(r)}{\theta(r)},$$

$$\frac{\phi_1\phi_2\theta(q)}{\theta(q)} = \frac{\psi_1\psi_2\theta(r)}{\theta(r)} \quad \ldots\ldots (9),$$

$$\frac{\phi_1^2\phi_2\theta(q)}{\theta(q)} = \frac{\psi_1^2\psi_2\theta(r)}{\theta(r)}, \qquad \frac{\phi_1\phi_2^2\theta(q)}{\theta(q)} = \frac{\psi_1\psi_2^2\theta(r)}{\theta(r)},$$

$$\frac{\phi_1^3\theta(q)}{\theta(q)} = \frac{\psi_1^3\theta(r)}{\theta(r)}, \qquad \frac{\phi_2^3\theta(q)}{\theta(q)} = \frac{\psi_2^3\theta(r)}{\theta(r)}.$$

Of the above equations, the fifth, (9), will after development involve in its numerators the coefficients of all the

equations of the original system. By this circumstance it is distinguished from the remaining eight, which are in fact no other than would be obtained, by applying the theorem for binary systems to the separate pairs of equations which may be selected from (8). The development of (9) gives

$$\frac{\Sigma\{abc+2def-(ad^2+be^2+cf^2)\}}{abc+2def-(ad^2+be^2+cf^2)}=\frac{\Sigma(a'b'c'+\&c.)}{abc+\&c.},$$

where Σ, applied to any particular terms, denotes the aggregate of all similar combinations, which are of the first degree with respect to the coefficients of each of the primitive equations; thus,

$$\Sigma abc = ab_1c_2 + ab_2c_1 + a_1bc_2 + a_1b_2c + a_2bc_1 + a_2b_1c,$$

the remaining eight equations may be derived from (56), &c., Part I.

3. When the system (4) is linear, the theorem (6) ceases to be interpretable, and is replaced by the following. Let the values of x_1, x_2, ... x_m, in terms of the other set of variables, be

$$\left.\begin{aligned} x_1 &= \lambda_1 y_1 + \lambda_2 y_2 \ldots\ldots + \lambda_m y_m \\ x_2 &= \mu_1 y_1 + \mu_2 y_2 \ldots\ldots + \mu_m y_m \\ &\cdots \qquad \cdots \qquad \cdots \\ x_m &= \rho_1 y_1 + \rho_2 y_2 \ldots\ldots + \rho_m y_m \end{aligned}\right\} \ldots\ldots (10);$$

and let E be that function of the constants obtained by eliminating the variables from the second members of the above equations, expressed under their most general forms, and equated to 0. Let E_1 be a similar function of the constants involved in the first members of any m equations of the original system (4), and E_2 of those entering into the second members of the same equations, then

$$EE_1 = E_2 \ldots\ldots (11),$$

whence all the relations may be found.

4. Hitherto we have confined ourselves to the supposition, that the equations of a proposed system are all of the same degree. When such is not the case, recourse must be had to one of the following methods:—

First, We may raise by involution each of the primitive equations to the degree indicated by the least common multiple of the indices of their degrees, and then apply the theorems of the preceding sections.

Secondly, The second method depends on the judicious application of the principle which I am about to demonstrate; a principle the applications of which are not restricted to homogeneous functions, and which, if properly

followed out, may be found of service in more than one department of analysis.

Let U, a function of $x_1, x_2, \ldots x_m$, be linearly transformed into V, a function of $y_1, y_2, \ldots y_m$, by virtue of the relations (10).

Differentiating, we get

$$\left.\begin{aligned} dx_1 &= \lambda_1 dy_1 + \lambda_2 dy_2 \ldots + \lambda_m dy_m \\ dx_2 &= \mu_1 dy_1 + \mu_2 dy_2 \ldots + \mu_m dy_m \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \\ dx_m &= \rho_1 dy_1 + \rho_2 dy_2 \ldots + \rho_m dy_m \end{aligned}\right\} \ldots\ldots (12),$$

$$\frac{dU}{dx_1}dx_1 + \frac{dU}{dx_2}dx_2 \ldots + \frac{dU}{dx_m}dx_m = \frac{dV}{dy_1}dy_1 + \frac{dV}{dy_2}dy_2 \ldots + \frac{dV}{dy_m}dy_m \ldots (13),$$

$$\frac{d^2U}{dx_1^2}dx_1^2 + 2\frac{d^2U}{dx_1 dx_2}dx_1 dx_2 + \&c. = \frac{d^2V}{dy_1^2}dy_1^2 + \frac{d^2V}{dy_1 dy_2}dy_1 dy_2 + \&c. \ldots (14),$$

$$\Sigma K \frac{d^n U}{dx_1^a dx_2^b \ldots dx_m^c} dx_1^a dx_2^b \ldots dx_m^c$$

$$= \Sigma K \frac{d^n V}{dy_1^a dy_2^b \ldots dy_m^c} dy_1^a dy_2^b \ldots dy_m^c \ldots . (15).$$

Now since $x_1, x_2, \ldots x_m$, are independent, $dx_1, dx_2, \ldots dx_m$, are independent also. The only relations into which they enter are those of (12), by which they are linearly connected with $dy_1, dy_2, \ldots dy_m$, in precisely the same way as $x_1, x_2, \ldots x_m$, are connected with $y_1, y_2, \ldots y_m$. It is hence evident, that the second members of (13), (14), (15), may be regarded as formed from their respective first members, by the substitution of the values of $dx_1, dx_2, \ldots dx_m$, given in (12). Indeed the coefficients, $\frac{dU}{dx}, \frac{dU}{dx_2}$, &c. though variable as being functions of $x_1, x_2, \ldots x_m$, are nevertheless constant relatively to $dx_1, dx_2, \ldots, dx_m$. It is therefore clear that the equations (13), (14), (15), regarded as homogeneous with respect to the differentials, fulfil among their coefficients the same relations, and are subject to the same general laws as if those coefficients were absolutely constant.

By the application of this principle, it may be shewn that *the discussion of a given multiple system of equations may be reduced to that of another system, whose common index shall be equal to the greatest common measure of the indices of the original equations, or to any proposed multiple of that quantity.*

5. Given the binary system,

$$ax + by = a'x' + b'y' \ldots (16), \qquad q = r \ldots (17),$$

the latter equation being homogeneous, and of the n^{th} degree.

By differentiation,

$$\left.\begin{array}{c} adx + bdy = a'dx' + b'dy' \\ \dfrac{dq}{dx}\,dx + \dfrac{dq}{dy}\,dy = \dfrac{dr}{dx'}\,dx' + \dfrac{dr}{dy'}\,dy' \end{array}\right\} \ldots\ldots (18);$$

these equations being linear with respect to dx, dy, dx', dy', we have by (11),

$$E\left(a\frac{dq}{dy} - b\frac{dq}{dx}\right) = a'\frac{dr}{dy'} - b'\frac{dr}{dx'},$$

which may be written under the form,

$$E\left(a\frac{d}{dy} - b\frac{d}{dx}\right)q = \left(a'\frac{d}{dy'} - b'\frac{d}{dx'}\right)r,$$

and would give, if the differentiations were effected, a homogeneous equation of the $(n-1)^{\text{th}}$ degree. This we shall represent by $q' = r'$. Again, therefore, applying the theorem (11), we obtain

$$E\left(a\frac{d}{dy} - b\frac{d}{dx}\right)q' = \left(a'\frac{d}{dy'} - b'\frac{d}{dx'}\right)r',$$

or substituting for q' and r', the respective members for which they stand,

$$E^2\left(a\frac{d}{dy} - b\frac{d}{dx}\right)^2 q = \left(a'\frac{d}{dy'} - b'\frac{d}{dx'}\right)^2 r,$$

and thus finally, after η repetitions of the process,

$$E^{\eta}\left(a\frac{d}{dy} - b\frac{d}{dx}\right)^{\eta} q = \left(a'\frac{d}{dy'} - b'\frac{d}{dx'}\right)^{\eta} r \ldots .(19).$$

It remains to determine E. Now by (87), Part I.,

$$\theta(q) = \frac{\theta(r)}{E^{\frac{\gamma n}{m}}} = \frac{\theta(r)}{E^{n(n-1)}} \ldots\ldots (20),$$

for when $m = 2$, it is easily seen that $\gamma = 2(n-1)$. And since the relations among the variables are the same as those among the differentials, it is evident that the value of E, determined from (20), may be applied in the present case. Substituting that value in (19), we obtain

$$\frac{\left(a\dfrac{d}{dy} - b\dfrac{d}{dx}\right)^{\eta}}{\{\theta(q)\}^{\frac{\eta}{n(n-1)}}}\,q = \frac{\left(a'\dfrac{d}{dy'} - b'\dfrac{d}{dx'}\right)^{\eta}}{\{\theta(r)\}^{\frac{\eta}{n(n-1)}}}\,r \ldots .(21).$$

By giving to η the values $1, 2, 3 \ldots n$, we shall obtain a series of homogeneous equations, of which the last but one will, with

(16), determine the linear system, and of which the last will give a relation among the constants. The remaining relations may be deduced, in various ways, from the remaining equations of the above system.

Ex. 1. Suppose the primitive equations to be

$$ax + by = a'x' + b'y' \ldots\ldots\ldots\ldots\ldots (22),$$

$$Ax^2 + 2Bxy + Cy^2 = A'x'^2 + 2B'x'y' + C'y'^2 \ldots\ldots (23).$$

Here $q = Ax^2 + 2Bxy + Cy^2$, $\theta(q) = AC - B^2$, $n = 2$, &c.; substituting in (21) and making $\eta = 1$, we have

$$\frac{a(Bx + Cy) - b(Ax + By)}{\sqrt{(AC - B^2)}} = \frac{a'(B'x' + C'y') - b'(A'x' + B'y')}{\sqrt{(AC - B^2)}},$$

$$\text{or} \quad \frac{(aB - bA)x + (aC - bB)y}{\sqrt{(AC - B^2)}} = \frac{(a'B' - b'A')x' + (a'C' - b'B')y'}{\sqrt{(AC - B^2)}} \ldots (24);$$

this completes the linear system. Again, making $\eta = 2$, we find

$$\frac{a^2C - 2abB + b^2A}{AC - B^2} = \frac{a'^2C' - 2a'b'B' + c'^2A'}{A'C' - B'^2} \ldots\ldots (25),$$

which expresses the relation among the constants, and may be easily verified by the former method.

We may here observe, that the number of the relations among the constants of a proposed system of homogeneous equations, will be equal to the excess of the number of constants in the functions to be transformed, supposed to be expressed under their most general types, over the square of the number of variables in those functions. This is evident on the common principles of elimination, for the constants in the linear theorems (10), are in number equal to the square of the number of the variables, and it is by an implied elimination of these, that we arrive at the constant relations sought. This rule fails when there are not sufficient data to render the linear system determinate, as in the next example but one; whether in any other case, I have not determined.

Ex. 2. Let the primitive equations be

$$y = mx' + ny' \ldots\ldots (26),$$

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3 = a'x'^3 + d'y'^3 \ldots\ldots (27).$$

Here $\theta(q) = (ad - bc)^2 - 4(b^2 - ac)(c^2 - bd)$, $\theta(r) = a'^2d'^2$, and by (21), the signs being changed for convenience,

$$\frac{\left(\dfrac{d}{dx}\right)^\eta}{\{\theta(q)\}^{\frac{\eta}{6}}} q = \frac{\left(n\dfrac{d}{dx'} - m\dfrac{d}{dy'}\right)^\eta}{\{\theta(r)\}^{\frac{\eta}{6}}} r.$$

Giving to η the values 1, 2, 3, and performing the differentiations, we have

$$\frac{ax^2 + 2bxy + cy^2}{\{\theta(q)\}^{\frac{1}{6}}} = \frac{na'x'^2 - md'\,y'^2}{(a'd')^{\frac{1}{3}}} \ldots\ldots\ldots (28),$$

$$\frac{ax + by}{\{\theta(q)\}^{\frac{1}{3}}} = \frac{n^2a'x' + m^2d'y'}{(a'd')^{\frac{2}{3}}} \ldots\ldots\ldots (29),$$

$$\frac{a}{\{\theta(q)\}^{\frac{1}{2}}} = \frac{n^3a' - m^3d'}{a'd'} = \frac{n^3}{d'} - \frac{m^3}{a'} \ldots\ldots (30).$$

From (28) and (29), by (25) we have

$$\frac{b^2 - ac}{\{\theta(q)\}^{\frac{2}{3}}} = \frac{mn}{(a'd')^{\frac{1}{3}}} \ldots\ldots\ldots\ldots\ldots\ldots (31);$$

of the above results, (30) and (31) determine the relations among the constants, and (29) completes the linear system.

We will now examine the forms of solution developed by the principle of our first method; for this purpose cubing (26) our equations become

$$y^3 = m^3x'^3 + 3m^2nx'^2y' + 3mn^2x'y'^2 + n^3y'^3,$$

$$ax^3 - 3bx^2y + 3cxy^2 + dy^3 = a'x'^3 + 3b'x'^2y' + 3c'x'y'^2 + d'y'^3,$$

b' and c' being supposed to vanish *after the differentiations*, and the symbolical formula for binary systems gives

$$\frac{\left(\frac{d}{d.d}\right)^\eta \theta(q)}{\theta(q)} = \frac{\left(m^3\,\frac{d}{da'} + m^2n\,\frac{d}{db'} + mn^2\,\frac{d}{dc'} + n^3\,\frac{d}{dd'}\right)^\eta \theta(r)}{\theta(r)}\,.$$

The first member I shall not develope; in the second we have

$\theta(r) = (a'd' - b'c')^2 - \&c. = a'^2d'^2 - 6a'b'c'd' - 3b'^2c'^2 + 4b'^3d' + 4c'^3a'$,

giving to η the values 1 and 2, and putting for brevity

$$\theta(q) = \theta,\ \frac{d\theta(q)}{dd} = \theta',\ \frac{d^2\theta(q)}{dd^2} = \theta'',\ \text{we obtain}$$

$$\frac{\theta'}{\theta} = \frac{2m^3a'd'^2 + 2n^3a'^2d'}{a'^2d'^2},$$

$$\frac{\theta''}{\theta} = \frac{2m^6d'^2 + 8m^3n^3a'd' + 2n^6a'^2 - 12m^3n^3a'd'}{a'^2d'^2},$$

whence, by reduction,

$$\left.\begin{aligned} \frac{m^3}{a'} + \frac{n^3}{d'} &= \frac{\theta'}{2\theta} \\ \frac{m^3}{a'} - \frac{n^3}{d'} &= \sqrt{\frac{\theta''}{2\theta}} \end{aligned}\right\} \ldots\ldots (32).$$

If $\eta > 2$, the theorem gives $0 = 0$, whence there are no other relations than the above. That they are equivalent to (30) and (31), I have confirmed by actual examination.

The above is a very instructive example. The intelligent reader will observe, that while the method employed in the former of the two solutions, possesses in every other respect the advantage, it is in this particular deficient, that it does not sufficiently limit the number of the final relations; for by employing (29) in the room of (26), or by various other strictly legitimate artifices, we might extend to infinity the series of the constant relations, of which it may however be demonstrated, that two only are independent. This peculiarity will again fall under our notice.

6. As examples of equations with three variables, let us take

Ex. 1. $$ax + by + cz = a'x' + b'y' + c'z' \;..\;(33),$$
$$Ax^2 + By^2 + Cz^2 + 2Dyz + 2Exz + 2Fxy = A'x'^2 + \&c. \;......\;(34).$$

Here the second method is inapplicable, by the first we get

$$\frac{a^2L + b^2M + c^2N + bcS + acT + abU}{ABC + 2DEF - (AD^2 + BE^2 + CF^2)} = \frac{a'^2L' + \&c.}{A'B'C' + \&c.} \;.....\;(35),$$

L, M, N, &c. as in (57), Part I.

Ex. 2. $$\left.\begin{array}{r} ax + by + cz = a'x' + b'y' + c'z' \\ a_1x + b_1y + c_1z = a_1'x' + b_1'y' + c_1'z' \\ x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2 \end{array}\right\} \;.....\;(36);$$

uniting the two methods, we obtain for the constant relations,

$$\left.\begin{array}{r} a^2 + b^2 + c^2 = a'^2 + b'^2 + c'^2 \\ a_1^2 + b_1^2 + c_1^2 = a_1'^2 + b_1'^2 + c_1'^2 \\ aa_1 + bb_1 + cc_1 = a'a_1' + b'b_1' + c'c_1' \end{array}\right\} \;.....\;(37),$$

and to complete the linear system,

$$(bc_1 - b_1c)x + (ca_1 - c_1a)y + (ab_1 - a_1b)z = (b'c_1' - b_1'c')x' + \&c. \;.\;(38).$$

The above, which is a very simple case, is merely given to shew the wide range of the method.

7. Let us next take the general binary system,

$$\Sigma\, ka_1 x_1^{\alpha} x_2^{\beta} \ldots x_m^{\mu} = \Sigma\, kb_1 y_1^{\alpha} y_2^{\beta} \ldots y_m^{\mu} \text{ for } q = r \;....\;(39),$$
$$Q = R \;......................\;(40),$$

the former equation being of the n^{th}, the latter of the m^{th} degree, supposing $m > n$ and $n > 1$.

The general formula of reduction will be found to be

$$\frac{\left\{\Sigma\left(\frac{d\,\theta(q)}{da_1}\;\frac{d^n}{dx_1{}^\alpha dx_2{}^\beta \ldots dx^\mu{}_m}\right)\right\}^\eta}{\{\theta(q)\}^\eta}\,Q$$

$$=\frac{\left\{\Sigma\left(\frac{d\,\theta(r)}{db_1}\;\frac{d^n}{dy_1{}^\alpha dy_2{}^\beta \ldots dy^\mu{}_m}\right)\right\}^\eta}{\{\theta(r)\}^\eta}\,R\,..(41);$$

but except when the equations are of a very elevated degree, it will perhaps be more simple to employ directly the general principles of § 4.

Ex. 1. Given the system,

$$ax^2+2bxy+cy^2=a'x'^2+2b'x'y'+c'y'^3\;\ldots\ldots\ldots(42),$$

$$Ax^3+3Bx^2y+3Cxy^2+Dy^3=A'x'^3+3B'x'^2y'+3C'x'y'^2+D'y'^3\ldots(43).$$

taking the second differentials, we have

$$adx^2+2bdxdy+cdy^2=a'dx'^2+2b'dx'dy'+c'dy'^2,$$

$$(Ax+By)dx^2+2(Bx+Cy)dxdy+(Cx+Dy)dy^2=(A'x'+B'y')dx'^2+\&c.$$

and treating these as homogeneous equations of the second degree relatively to dx, dy, dx', dy', we obtain

$$\frac{(Ax+By)(Cx+Dy)-(Bx+Cy)^2}{ac-b^2}$$

$$=\frac{(A'x'+B'y')(C'x'+D'y')-(B'x'+C'y')^2}{a'c'-b'^2}$$

$$\frac{a(Cx+Dy)-2b(Bx+Cy)+c(Ax+By)}{ac-b^2}=\frac{a'(C'x'+D'y')-\&c.}{a'c'-b'^2},$$

or arranging with reference to x and y, x' and y',

$$\frac{(AC-B^2)x^2-2(AD-BC)xy+(BD-C^2)y^2}{ac-b^2}=\frac{(A'C'-B'^2)x'^2-\&c.}{a'c'-b'^2}$$

$$\frac{(aC-2bB+cA)x+(aD-2bC+cB)y}{ac-b^2}=\frac{(a'C'-2b'B'+c'A')x'+\&c.}{a'c'-b'^2}$$

Let $AC-B^2=p$. $AD-BC=q$. $BD-C^2=r$.
$aC-2bB+cA=s$. $aD-2bC+cB=t$.
$A'C'-B'^2=p'$. &c. &c.

then have we the following system of equations,

$$ax^2+2bxy+cy^2=a'x'^2+2b'x'y'+c'y'^2\ldots\ldots(44),$$

$$\frac{px^2-qxy+ry^2}{ac-b^2}=\frac{p'x'^2-q'x'y'+r'y'^2}{a'c'-b'^2}\ldots\ldots(45).$$

$$\frac{sx+ty}{ac-b^2} = \frac{s'x'+t'y'}{a'c'-b'^2} \quad \ldots . (46).$$

From (44) and (45), which are homogeneous of the second degree,

$$\frac{q^2-4pr}{(ac-b^2)^3} = \frac{q'^2-4p'r'}{(a'c'-b'^2)^3} \quad \ldots . (47),$$

$$\frac{ar-bq+cp}{(ac-b^2)^2} = \frac{a'r'-b'q'+c'p'}{(a'c'-b'^2)^2} \quad \ldots . (48).$$

Also, from (44) and (46), *vide* Ex. 1, §. 5,

$$\frac{at^2-2bst+cs^2}{(ac-b^2)^3} = \frac{a'b'^2-2b's't'+c's'^2}{(a'c'-b'^2)^3} \quad \ldots . (49),$$

$$\frac{(at-bs)\,x+(bt-cs)\,y}{(ac-b^2)^{\frac{3}{2}}} = \frac{(a't'-b's')\,x'+(b't'-c's')\,y'}{(a'c'-b'^2)^{\frac{3}{2}}} \quad \ldots . (50).$$

Of the above results (46) and (50) determine the linear system, (47) (48) and (49) express the relations among the constants. Other forms of solution may be found in infinite variety; thus a relation may be found from (45) and (46), but they will all be combinations of those we have already obtained. Meanwhile there is nothing in the process which appears to indicate when it is necessary to stop, and what is the nature of that functional connexion which must exist among the interminable series of equations, to which, if continued, it would give birth. To the discussion of this question, I would especially direct the attention of those who may be disposed to take up the subject.

8. Linear transformations have hitherto been chiefly applied to the purpose of taking away from a proposed homogeneous function, those terms which involve the products of the variables. It may be observed that this problem resolves itself into two principal cases: the first is that in which the transformations, besides being linear, are understood to represent a geometrical change of axes, or are such as to involve an obvious extension of this analogy; the second case is when no other condition than that of linearity is introduced. It is to the former of the above cases, and to that only as developed in the first of the subjoined examples, that the efforts of analysts appear to have been hitherto directed.

Ex. 1. To transform the homogeneous function Q, of the second degree, with m variables, $x_1, x_2, \ldots . x_m$, to the form $B_1y_1^2 + B_2y_2^2 \ldots . + B_my_m^2$, conformably with the condition

$$x_1^2 + x_2^2 \ldots . + x_m^2 = y_1^2 + y_2^2 \ldots . + y_m^2.$$

Representing the above condition by $q = r$, form the equation $\theta(Q + hq) = 0$; the values of h, taken negatively, will determine $B_1, B_2, \ldots B_m$.

Ex. 2. To exhibit, under a general theorem, all the linear systems by which the function $ax^2 + 2bxy + cy^2$, may be reduced to the form $a'x'^2 + c'y'^2$, a' and c' being given.

Let m and n be any two quantities satisfying the condition

$$\frac{m^2}{a'} + \frac{n^2}{c'} = \frac{a}{ac - b^2} \ldots (51);$$

then are the general linear forms in question

$$\left.\begin{array}{c} y = mx' + ny' \\ \dfrac{ax + by}{\sqrt{(ac - b^2)}} = \dfrac{na'x' - mc'y'}{\sqrt{(a'c')}} \end{array}\right\} \ldots\ldots(52);$$

these results are deduced from Ex. 1, § 5.

Ex. 3. To take away the products of the variables from a proposed homogeneous function, of the form

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3,$$

by transformations equivalent to a geometrical change of axes.

Let θ represent the inclination of the axes x and y, θ' that of the unknown axes x' and y', and $a'x'^3 + d'y'^3$ the transformed function; then, by the aid of ten subsidiary quantities $p, q, r, s, t, P, Q, R, S, T$, the solution will be conveniently expressed in the annexed forms,

$$\left.\begin{array}{c} p = ac - b^2,\ q = ad - bc,\ r = bd - c^2, \\ s = a - 2b\cos\theta + c,\ t = b - 2c\cos\theta + d, \\ P = \sqrt{(q^2 - 4pr)},\ Q = p - q\cos\theta + r,\ R = s^2 - 2st\cos\theta + t^2. \\ \sin\theta' = \dfrac{P\sin\theta}{\sqrt{\{P^2(\sin\theta)^2 + Q^2\}}}, \\ S = R\left(\dfrac{\sin\theta'}{\sin\theta}\right)^6 - 2Q\left(\dfrac{\sin\theta'}{\sin\theta}\right)^4 + 2P\left(\dfrac{\sin\theta'}{\sin\theta}\right)^3, \\ T = R\left(\dfrac{\sin\theta'}{\sin\theta}\right)^6 - 2Q\left(\dfrac{\sin\theta'}{\sin\theta}\right)^4 - 2P\left(\dfrac{\sin\theta'}{\sin\theta}\right)^3, \\ a' = \dfrac{\sqrt{S} + \sqrt{T}}{2},\ d' = \dfrac{\sqrt{S} - \sqrt{T}}{2}, \end{array}\right\} \ldots(53);$$

and for the linear relations,

$$\left.\begin{array}{c} \dfrac{sx + ty}{(\sin\theta)^2} = \dfrac{a'x' + d'y'}{(\sin\theta')^2}, \\ \dfrac{(t - s\cos\theta)x - (s - t\cos\theta)y}{(\sin\theta)^3} = \dfrac{(d' - a'\cos\theta')x' - (a' - d'\cos\theta')y'}{(\sin\theta')^3} \end{array}\right\} \ldots(54)$$

When the original axes are rectangular, the above results are greatly simplified. The reader will perceive that they are deduced from Ex. 1, § 7.

We take as a numerical illustration the equation

$$37x^3 - 36x^2y - 36xy^2 + 37y^3 = 1,$$

the axes x and y being rectangular.

Here $a = 37$, $b = -12$, $c = -12$, $d = 37$. The resulting form is found to be

$$(\tfrac{7}{5})^3 x'^3 + (\tfrac{7}{5})^3 y'^3 = 1,$$

with the linear relations

$$x = \tfrac{4}{5}x' + \tfrac{3}{5}y',$$
$$y = \tfrac{3}{5}x' + \tfrac{4}{5}y'.$$

Ex. 4. To transform the function, $ax^3 + 3bx^2y + 3cxy^2 + dy^3$, to the form $a'x'^3 + d'y'^3$, a' and d' being given, and the transformation unrestricted by any other condition than that of linearity.

The direct solution of this problem is contained in the formulæ of Ex. 2, § 5, which shew that if m and n be so determined as to satisfy the conditions,

$$\frac{m^3}{a'} + \frac{n^3}{d'} = \frac{\theta'}{2\theta}, \quad \frac{m^3}{a'} - \frac{n^3}{d'} = \sqrt{\frac{\theta'}{2\theta}} \;\ldots\ldots\; (55);$$

then the linear system in question will be

$$\left.\begin{array}{c} y = mx' + ny' \\ \dfrac{ax + by}{(\theta)^{\frac{1}{3}}} = \dfrac{n^2a'x' + m^2d'y'}{(a'd')^{\frac{2}{3}}} \end{array}\right\} \ldots\ldots\; (56).$$

9. The doctrine of linear transformations may be elegantly applied to the solution of algebraic equations. The following example, in which I shall apply the above theorems, will clearly shew the nature of the connexion.

Ex. The most general form of the cubic equation is

$$av^3 + 3bv^2 + 3cv + d = 0 \;\ldots\; (57);$$

the simplest of possible forms is

$$v'^3 - 1 = 0 \;\ldots\ldots\; (58),$$

giving $v' = 1$. If by linear transformation we can reduce (57) to (58), the solution of the former will be derived from that of the latter. To effect this we first render them homogeneous by putting $v = \dfrac{x}{y}$, $v' = \dfrac{x'}{y'}$. The problem is then reduced to the discussion of the equation

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3 = x'^3 - y'^3 \;\ldots\ldots\; (59).$$

Here, since $a' = 1$, $d' = -1$, we have, by the theorems above,

$$m^3 - n^3 = \frac{\theta'}{2\theta};\ m^3 + n^3 = \sqrt{\left(\frac{\theta''}{2\theta}\right)} \ \ldots\ldots\ (60),$$

$$y = mx' + ny' \ \ldots\ldots\ (61),$$

$$\frac{ax + by}{\sqrt[3]{\theta}} = n^2x' - m^2y' \ \ldots\ldots\ (62).$$

From (60) we find

$$m = \sqrt[3]{\left\{\frac{\theta' + \sqrt{(2\theta\theta'')}}{4\theta}\right\}} \quad n = \sqrt{\left\{\frac{-\theta' + \sqrt{(2\theta\theta'')}}{4\theta}\right\}}.$$

Dividing (62) by (61), and making $\dfrac{x}{y} = v$, $\dfrac{x'}{y'} = v' = 1$, we find

$$\frac{av + b}{\sqrt[3]{\theta}} = \frac{n^2 - m^2}{n + m} = n - m;$$

$$\therefore\ av + b = \sqrt{\left\{\frac{-\theta' + \sqrt{(2\theta\theta'')}}{4}\right\}} + \sqrt[3]{\left\{\frac{-\theta' - \sqrt{(2\theta\theta'')}}{4}\right\}} \ldots (63),$$

in which it is only necessary to observe, that

$$\theta = \theta(q) = (ad - bc)^2 - 4(b^2 - ac)(c^2 - bd),$$

$$\theta' = \frac{d\,\theta(q)}{dd} = 2(a^2d - 3abc + 2b^3), \quad \theta'' = \frac{d^2\,\theta(q)}{dd^2} = 2a^2.$$

To extend this investigation to the equations of the fourth and fifth degree, will require the previous determination of $\theta(q)$ for those cases, a question tedious but not difficult, and to which either the method described in Part I., or the ingenious modes of elimination devised by Professor Sylvester, may be applied. As this question is of fundamental importance, and needs to be determined but once, it is much to be desired that some one, possessed of leisure, would undertake its discussion.

An equally important subject of inquiry presents itself in the connexion between linear transformations and an extensive class of theorems depending on partial differentials, particularly such as are met with in Analytical Geometry. It is not my intention to enter into the subject in this place, nor have I leisure either to pursue the inquiry, or to elucidate my present views in a separate paper. To those who may be disposed to engage in the investigation, it will, I believe, present an ample field of research and discovery. It is almost needless to observe, that any additional light which may be thrown on the general theory, and especially as respects the properties of the function $\theta(q)$, will tend to facilitate our further progress, and to extend the range of useful applications.

Lincoln, October 21st, 1841.

IV.—A METHOD OF OBTAINING ANY ROOT OF A NUMBER IN THE FORM OF A CONTINUED FRACTION.*

THE principle of the following method of approximating to any root of a number, will be best exhibited by taking first the simplest case, that of the square root.

Let N be the number, and let $N = a^2 + b$, a^2 being the square number next less than N. Then identically, $\sqrt{(N)} = a + \sqrt{(a^2 + b)} - a$. Now $\sqrt{(a^2 + b)} - a$ is less than unity, and may therefore be assumed equal to the continued fraction

$$\frac{1}{p+}\,\frac{1}{q+}\,\frac{1}{r + \&c.}\,.$$

Hence $\sqrt{(a^2 + b)} - a$ is less than $\frac{1}{p}$ and greater than $\frac{1}{p+1}$, so that p is the greatest whole number that satisfies the inequality

$$\sqrt{(a^2 + b)} - a < \frac{1}{p}$$

By squaring, $a^2 + b < a^2 + \frac{2a}{p} + \frac{1}{p^2}$; or $bp^2 - 2ap < 1$; whence p is readily found by trial. Again, $\sqrt{(a^2 + b)} - a$ is greater than $\dfrac{1}{p + \frac{1}{q}}$ and less than $\dfrac{1}{p + \frac{1}{q+1}}$. Hence q is the greatest whole number that satisfies the inequality $\sqrt{(a^2 + b)} - a > \dfrac{1}{p + \frac{1}{q}}$; or $b\left(p + \frac{1}{q}\right)^2 - 2a\left(p + \frac{1}{q}\right) > 1$. By means of this inequality, q may be found by trial when p is known. It is evident that the successive inequalities may be formed by substituting $p + \frac{1}{q}$ for p in the first, $q + \frac{1}{r}$ for q in the second, and so on, and changing the sign $>$ or $<$ at each substitution. This consideration will facilitate the numerical calculation, as will be seen by an example.

Let $N = 11$. Then $a = 3$, $b = 2$, and the first inequality is $2p^2 - 6p < 1$. Hence $p = 3$. The second inequality is therefore

* From a Correspondent.

$2\left(3+\frac{1}{q}\right)^2 - 6\left(3+\frac{1}{q}\right) > 1$, which gives $q^2 - 6q < 2$; hence $q = 6$. The third inequality is $\left(6+\frac{1}{r}\right)^2 - 6\left(6+\frac{1}{r}\right) > 2$, which gives $2r^2 - 6r < 1$. As this is the same as the first, the operations will recur, and we therefore have

$$\sqrt{(11)} = 3 + \frac{1}{3+}\ \frac{1}{6+}\ \frac{1}{3+\&c.}.$$

The way of proceeding in approximating to the cube root of a number is precisely analogous to that above. Let $N = a^3 + b$; then, as before, $\sqrt[3]{(a^3+b)} - a < \frac{1}{p}$. Hence $a^3 + b < a^3 + \frac{3a^2}{p} + \frac{3a}{p^2} + \frac{1}{p^3}$; or, $bp^3 - 3a^2p^2 - 3ap < 1$; and p is the greatest whole number that satisfies this inequality. Let, for example, $N = 10$; then $a = 2$, $b = 2$, and $2p^3 - 12p^2 - 6p < 1$. Hence $p = 6$. By substituting $6 + \frac{1}{q}$ for p in the above inequality, and changing $<$ into $>$, it will be found that $37q^3 - 66q^2 - 24q < 2$. Hence $q = 2$. The next inequality is $18r^3 - 156r^2 - 156r < 37$, from which $r = 9$, and so on. Hence approximately

$$\sqrt[3]{(10)} = 2 + \frac{1}{6+}\ \frac{1}{2+}\ \frac{1}{9} = \frac{265}{123};$$

this result is true to four places of decimals.

The same method applied in extracting the fourth root of 20, gives the approximate value

$$2 + \frac{1}{8+}\ \frac{1}{1+}\ \frac{1}{2+}\ \frac{1}{1+}\ \frac{1}{1} = \frac{129}{61},$$

which is very nearly true to five places of decimals; the next quotient is found to be 22.

It is evident that this process may be employed to approximate to any root of a fraction. Let, for instance,

$$\sqrt{\left(\frac{30}{7}\right)} = 2 + \frac{1}{p+}\ \frac{1}{q+\&c.};\ \text{then}\ \sqrt{\left(\frac{30}{7}\right)} - 2 < \frac{1}{p}.$$

Whence $2p^2 - 28p < 7$, and $p = 14$. The second inequality is therefore $2\left(14+\frac{1}{q}\right)^2 - 28\left(14+\frac{1}{q}\right) > 7$, which gives $7q^2 - 28q < 2$. Hence $q = 4$. The third is, $2r^2 - 28r < 7$, and is the same as the first. Consequently

$$\sqrt{\left(\frac{30}{7}\right)} = 2 + \frac{1}{14+}\ \frac{1}{2+}\ \frac{1}{14+}\ \frac{1}{2+\&c.},$$

a result easily verified by obtaining in the usual manner the value of this recurring continued fraction.

Let it be required to approximate to $\sqrt[3]{(\frac{2}{3})}$ in a continued fraction. As this quantity is less than unity we may assume

$$\sqrt[3]{\left(\frac{2}{3}\right)} = \frac{1}{p+}\,\frac{1}{q+}\,\frac{1}{r+\&c.}\,.$$

Hence p is the greatest whole number that satisfies the condition $\sqrt[3]{\left(\frac{2}{3}\right)} < \frac{1}{p}$. The successive inequalities and the resulting values of p, q, r, &c. will be found to be as follows:—

$$\begin{array}{lll} 2p^3 & < 3, & \therefore p = 1 \\ q^3 - 6q^2 - 6q & < 2, & \therefore q = 6 \\ 38r^3 - 30r^2 - 12r & < 1, & \therefore r = 1 \\ 5s^3 - 42s^2 - 84s & < 38, & \therefore s = 10. \end{array}$$

Hence $\sqrt[3]{\left(\frac{2}{3}\right)} = \frac{1}{1+}\,\frac{1}{6+}\,\frac{1}{1+}\,\frac{1}{10+} = \frac{76}{87}$ nearly; this value is correct to four places of decimals.

It appears therefore that the method here proposed gives the means of throwing any root of either an integral or a fractional quantity into the form of a continued fraction, and consequently of approximating to the root with any required degree of accuracy.

C.

V.—NOTES ON SOME POINTS IN FORMAL OPTICS.*

1. *Construction for the place of the primary focal line after oblique reflexion at a spherical surface.*

Let E be the centre of the surface, P the focus, PR the axis of an incident pencil meeting the surface in R, RQ the axis of the reflected pencil. From E draw EF perpendicular to PR meeting PR in F; from F draw FG perpendicular to ER meeting ER in G, and draw the straight line PG meeting RQ in Q. Then Q will be the place of the primary focal line. For by the construction Q would be the place of the secondary focal line, after reflexion at R of a pencil having P for its focus, from a surface having G for its centre. Therefore, if

$$ER = r,\ PRE = \phi,\ PR = u,\ QR = v,\ \frac{1}{v} + \frac{1}{u} = \frac{2}{RG}\cos\phi.$$

But $RG = r\,(\cos\phi)^2$, $\therefore \frac{1}{v} + \frac{1}{u} = \frac{2}{r\cos\phi}$, $\therefore Q$ is the place of the primary focal line.

* From a Correspondent.

2. *Construction for the place of the primary focal line after oblique refraction at a spherical surface.*

Let E be the centre of the surface, P the focus, PR the axis of an incident pencil meeting the surface in R, RQ the axis of the refracted pencil. From P draw PF perpendicular to PR meeting RE in F; from F draw FG perpendicular to RE meeting PR in G; draw the straight line EG meeting RQ in K; from K draw KH perpendicular to RE meeting RE in H; from H draw HQ perpendicular to RQ meeting RQ in Q. Then Q will be the place of the primary focal line. For by the construction Q would be the place of the secondary focal line of a pencil having G for its focus, and GR for its axis after refraction at R. Therefore, if $RE = r$, $PRE = \phi$, $QRE = \phi'$, $PR = u$, $QR = v$, $\sin\phi = \mu \sin\phi'$,

$$\frac{\mu}{KR} - \frac{1}{GR} = \frac{1}{r}(\mu\cos\phi' - \cos\phi),\ u = GR(\cos\phi)^2, v = KR(\cos\phi')^2.$$

Therefore $$\frac{\mu(\cos\phi')^2}{v} - \frac{(\cos\phi)^2}{u} = \frac{1}{r}(\mu\cos\phi' - \cos\phi).$$

Therefore Q is the place of the primary focal line.

3. *Construction for the place of the primary focal line after oblique refraction at a plane surface.*

Let P be the focus of the incident pencil, PR its axis meeting the surface in R; RQ the axis of the refracted pencil. Draw RF perpendicular to the surface; draw PF perpendicular to PR meeting RF in F; draw FG perpendicular to RF meeting RP in G; through G draw GK parallel to RF meeting RQ in K; draw KH perpendicular to RF meeting RF in F, and HQ perpendicular to RQ meeting RQ in Q. Then Q will be the place of the primary focal line. For if $PRF = \phi$, $QRF = \phi'$, $\sin\phi = \mu\sin\phi'$, $PR = u$, $QR = v$, $KR = \mu GR$, $u = GR(\cos\phi)^2$, $v = KR(\cos\phi')^2$.

Therefore $$\mu\frac{(\cos\phi')^2}{v} - \frac{(\cos\phi)^2}{u} = 0.$$

Therefore Q is the place of the primary focal line.

W. H. M.

VI.—ON ELLIPTIC FUNCTIONS.

By B. Bronwin.

Sir James Ivory has greatly simplified the Theory of Elliptic Functions as given by M. Jacobi. He has also applied the theory to the case where the index of multiplica-

tion is an even number. But the case of which M. Jacobi has treated is susceptible of still further simplification.

The following auxiliary formulæ are from Jacobi, page 32.

Put $am\,.\,u = a$, $am\,.\,v = b$, $am\,.\,(u+v) = \sigma$, $am\,.\,(u-v) = \theta$; then if $\Delta a = \sqrt{(1-k^2\sin^2 a)}$, $\Delta b = \sqrt{(1-k^2\sin^2 b)}$, $k^2 + k'^2 = 1$,

$$\left.\begin{aligned}
\sin\sigma &= \frac{\sin a\cos b\Delta b + \sin b\cos a\,\Delta a}{1-k^2\sin^2 a\sin^2 b}\\
\cos\sigma &= \frac{\cos a\cos b - \sin a\sin b\,\Delta a\Delta b}{1-k^2\sin^2 a\sin^2 b}\\
\Delta\sigma &= \frac{\Delta a\Delta b - k^2\sin a\sin b\cos a\cos b}{1-k^2\sin^2 a\sin^2 b}\\
\sin\theta &= \frac{\sin a\cos b\Delta b - \sin b\cos a\Delta a}{1-k^2\sin^2 a\sin^2 b}\\
\cos\theta &= \frac{\cos a\cos b + \sin a\sin b\Delta a\Delta b}{1-k^2\sin^2 a\sin^2 b}\\
\Delta\theta &= \frac{\Delta a\Delta b + k^2\sin a\sin b\cos a\cos b}{1-k^2\sin^2 a\sin^2 b}\\
\sin\sigma\sin\theta &= \frac{\sin^2 a - \sin^2 b}{1-k^2\sin^2 a\sin^2 b}\\
\cos\sigma\cos\theta &= \frac{\cos^2 a\Delta^2 b - k'^2\sin^2 b}{1-k^2\sin^2 a\sin^2 b}\\
\Delta\sigma\Delta\theta &= \frac{\Delta^2 a\cos^2 b + k'^2\sin^2 b}{1-k^2\sin^2 a\sin^2 b}
\end{aligned}\right\}\ \ldots\ (A).$$

The transformation to be effected is

$$\frac{d\psi}{\sqrt{(1-h^2\sin^2\psi)}} = \frac{\beta d\phi}{\sqrt{(1-k^2\sin^2\phi)}},\ \text{or}\ dv = \beta du\ \ldots\ (B).$$

Here $\psi = am\,.\,v$, $\phi = am\,.\,u$. Let $v = H$ when $\psi = \frac{\pi}{2}$, $u = K$ when $\phi = \frac{\pi}{2}$, $h^2 + h'^2 = 1$; and let H' and K' be what H and K become when h is changed into h' and k into k'. Make $\omega = \frac{K}{n}$, n being an odd integer. Moreover when $u = 0$, ω, 2ω, 3ω, &c; suppose $v = 0$, H, $2H$, $3H$, &c. And to abridge as much as possible we shall put $s\,.\,a$ for sin amplitude, $c\,.\,a$ for cos am, and $t\,.\,a$ for tan am.

Assume

$$s.a.v = \frac{s.a.u\ s.a\,(u+2\omega)\,s.a\,(u+4\omega)\ldots s.a\,\{u+2\,(n-1)\,\omega\}}{s.a.\omega\ s.a.3\omega\ s.a.5\omega\ldots s.a\,(2n-1)\,\omega}\ \ldots(1),$$

$$c.a.v = \frac{c.a.u\, c.a\,(u+2\omega)\, c.a\,(u+4\omega) \ldots c.a\left\{u+2\,(n-1)\,\omega\right\}}{c.a.2\omega\; c.a.4\omega\; c.a.6\omega \ldots c.a.2\,(n-1)\,\omega} \ldots (2).$$

When u is changed into $u + 2\omega$, each factor of (1) and (2) goes into the succeeding one, and the last into the first with a contrary sign. The second members therefore remain unchanged except as to the sign. Consequently if $u = 0,\ 2\omega,\ 4\omega$, &c.; $\sin am\,.\,v = 0$; $\cos am\,.\,v = \pm 1$. If $u = \omega,\ 3\omega,\ 5\omega$, &c.; $\sin am\,.\,v = \pm 1$; $\cos am\,.\,v = 0$, because some factor in the numerator of (2) becomes $\cos am\,.\,n\omega = \cos am\,.\,K = 0$. The second members of (1) and (2) therefore are suitable expressions of the values of the first.

In (1) change u into $u + K$, v into $v + nH$; then by the first of (A) we have

$$\sin am\,.(v+nH) = \pm\,\frac{\cos am\,.\,v}{\Delta am\,.\,v},\ \sin am\,.(u+K) = \frac{\cos am\,.\,u}{\Delta am\,.\,u}.$$

The other factors will be similarly changed; and (1) becomes, putting A for its denominator,

$$\pm\,\frac{c\,.\,a\,.\,v}{\Delta\,.\,a\,.\,v} = \frac{c\,.\,a\,.\,uc\,.\,a\,(u+2\omega) \ldots c\,.\,a\left\{u+2\,(n-1)\,\omega\right\}}{A\Delta\,.\,a\,.\,u\Delta\,.\,a\,(u+2\omega) \ldots \Delta\,.\,a\left\{u+2\,(n-1)\,\omega\right\}}.$$

Multiplying this, member by member, by (1); we have

$$\pm\,A^2\,\frac{s.a.v\; c.a.v}{\Delta.a.v} = \frac{s.a.u\; c.a.u}{\Delta.a.u}\,.\,\frac{s.a\,(u+2\omega)\; c.a\,(u+2\omega)}{\Delta.a\,(u+2\omega)} \ldots (3).$$

This result is derived entirely from (1), and must therefore give the same relation between v and u which (1) does.

In (2) change u into $u + K$, v into $v + nH$. Then by the second of (A) we find

$$\cos am(v+nH) = \mp\,\frac{h'\sin am.v}{\Delta am.v},\ \cos am(u+K) = -\,\frac{k'\sin am.u}{\Delta am.u},\text{\&c.}$$

Thus (2) becomes, putting B for its denominator,

$$\pm\,\frac{h's.a.v}{\Delta.a.v} = \frac{k'^n s.a.us.a\,(u+2\omega) \ldots s.a.\left\{u+2\,(n-1)\,\omega\right\}}{B\Delta.a.u\Delta.a\,(u+2\omega) \ldots \Delta.a\left\{u+2\,(n-1)\,\omega\right\}}.$$

Multiplying this by (2), member by member; we obtain

$$\pm\,\frac{B^2h'}{k'^n}\,.\,\frac{s.a.vc.a.v}{\Delta.a.v} = \frac{s.a.uc.a.u}{\Delta.a.u}\,.\,\frac{s.a\,(u+2\omega)\; c.a\,(u+2\omega)}{\Delta.a\,(u+2\omega)} \ldots (4).$$

This last is derived entirely from (2), and must therefore give the same relation between v and u which (2) does. But if $A^2 = \dfrac{B^2h'}{k'^n}$, (4) is the same as (3). Therefore (1) and (2) give the same relation between v and u, or they are derivable from each other.

The equation $A^2 k'^n = B^2 h'$ gives

$$h' = k'^n \left\{ \frac{s.a.\omega \; s.a.3\omega. \dots s.a\,(2n-1)\,\omega}{c.a.2\omega \; c.a.4\omega \dots c.a\,2\,(n-1)\,\omega} \right\}^2$$

$$= k'^n \left\{ \frac{s.a.\omega \; s.a.3\omega. \dots s.a\,(n-2)\,\omega}{c.a.2\omega \; c.a.4\omega \dots c.a\,(n-1)\,\omega} \right\}^4$$

$$= \frac{k'^n}{\{\Delta\,(2\omega)\,\Delta\,(4\omega) \dots \Delta\,(n-1)\,\omega\}^4}.$$

If in (B) we make v and u infinitely small, we find

$$\beta = \frac{s.a.2\omega \; s.a.4\omega \dots s.a.2\,(n-1)\,\omega}{s.a.\omega \; s.a.3\omega \dots s.a\,(2n-1)\,\omega}$$

$$= \left\{ \frac{s.a.2\omega \; s.a.4\omega \dots s.a\,(n-1)\,\omega}{s.a.\omega \; s.a.3\omega \dots s.a\,(n-2)\,\omega} \right\}^2.$$

We notice here a particular property of (1). By (A) we find $s.a\,(u+2\omega)\,s.a\,\{u+2\,(n-1)\,\omega\} = s.a\,(2\omega+u)\,s.a\,(2\omega-u)$ $= \dfrac{s^2.a.2\omega - s^2.a.u}{1-k^2 s^2.a.2\omega \; s^2.a.u}$. If in this we change $s.a.u$ into $\dfrac{1}{ks.a.u}$, it becomes

$$\frac{1}{k^2}\cdot\frac{1-k^2 s^2.a.2\omega \; s^2.a.u}{s^2.a.2\omega - s^2.a.u} = \frac{1}{k^2}\cdot\frac{1}{s.a\,(u+2\omega)\,s.a\{u+(2-1)\,\omega\}}.$$

The same change would take place in every other corresponding pair of factors. Hence it is easily seen, that to change $s.a.u$ into $\dfrac{1}{ks.a.u}$, we should change $s.a.v$ into $\dfrac{1}{hs.a.v}$, if we make

$$h = k^n\,\{s.a.\omega\; s.a.3\omega \dots s.a\,(2n-1)\,\omega\}^2$$

$$= k^n\,\{s.a.\omega\; s.a.3\,\omega \dots s.a\,(n-2)\,\omega\}^4.$$

If in (2), we change $s.a.u$ into $\dfrac{1}{ks.a.u}$, $s.a.v$ into $\dfrac{1}{hs.a.v}$; and compare the result with a similar result before obtained; we shall see that h is the quantity so denominated in (B). From what has been done we easily derive the following:

$$\left.\begin{aligned}
s.a.v &= \sqrt{\left(\frac{k^n}{h}\right)}.\; s.a.u\; s.a\,(u+2\omega) \dots s.a\,\{u+2\,(n-1)\,\omega\} \\
c.a.v &= \sqrt{\left(\frac{h'k^n}{hk'^n}\right)}.\; c.a.u\; c.a\,(u+2\omega) \dots c.a\,\{u+2\,(n-1)\,\omega\}. \\
t.a.v &= \sqrt{\left(\frac{k'^n}{h'}\right)}.\; t.a.u\; t.a\,(u+2\omega) \dots t.a\,\{u+2\,(n-1)\,\omega\}. \\
\Delta.a.v &= \sqrt{\left(\frac{h'}{k'^n}\right)}.\; \Delta.a.u\; \Delta.a\,(u+2\omega) \dots \Delta.a\,\{u+2\,(n-1)\,\omega\}.
\end{aligned}\right\}(C).$$

If we make $x = s\,.\,a\,.\,u$; the first of (C), when developed, becomes $\sqrt{\left(\frac{h}{k^n}\right)}\,.\,s\,.\,a\,.\,v = xP\left(\frac{s^2\,.\,a\,.\,2r\omega - x^2}{1 - k^2x^2s^2\,.\,a\,.\,2r\omega}\right)$;

or $xP(x^2 - s^2\,.\,a\,.\,2r\omega) - \frac{h\beta}{k}s\,.\,a\,.\,vP\left(x^2 - \frac{1}{k^2s^2\,.\,a\,.\,2r\omega}\right) = 0$,

where P denotes the continued product, giving to r all the integer values from 1 to $\frac{n-1}{2}$ inclusive. The roots or values of x in this equation are

$$s\,.\,a\,.\,u,\ s\,.\,a(u+4\omega),\ s\,.\,a(u+8\omega) \ldots s\,.\,a\left\{u + 2(n-1)\omega\right\};$$
$$\text{and } -s\,.\,a(u+2\omega),\ -s\,.\,a(u+6\omega) \ldots -s\,.\,a\left\{u + 2(n-2)\omega\right\};$$
$$\text{or } s\,.\,a(u-4\omega),\ s\,.\,a(u-8\omega) \ldots s\,.\,a\left\{u - 2(n-1)\omega\right\}.$$

The coefficient of the second term of the above equation with its sign changed is equal to the sum of these roots. In like manner by making $x = c\,.\,a\,.\,u$, $x = t\,.\,a\,.\,u$, $x = \Delta\,.\,a\,.\,u$; we shall obtain similar results for these quantities. These results are

$$\left.\begin{aligned}
&\frac{h\beta}{k}\,.\,s.a.v = s.a.u + \Sigma\, s.a(u+4r\omega) + \Sigma\, s.a(u-4r\omega)\\
&\frac{h\beta}{k}\,.\,c.a.v = c.a.u + \Sigma\, c.a(u+4r\omega) + \Sigma\, c.a(u-4r\omega)\\
&\frac{h'\beta}{k'}\,.\,t.a.v = t.a.u + \Sigma\, t.a(u+2r\omega)\\
&\beta\,\Delta.a.v = \Delta.a.u + \Sigma\,\Delta.a(u+2r\omega)
\end{aligned}\right\} \ldots . (\mathrm{D}).$$

In the two first of (D) r is all the numbers $1, 2, \ldots \frac{n-1}{2}$; in the two last it is all the numbers $1, 2 \ldots n-1$.

By making $x = \frac{1}{s.a.u}$, $x = \frac{1}{c.a.u}$, &c. we might find a great many more formulæ similar to (D). But they may be more easily found from (D) by changing u into $u + K$, v into $v + nH$, or changing $s.a.u$ into $\frac{1}{k\,s.a.u}$, $s.a.v$ into $\frac{1}{h\,s.a.v}$; or by both these operations combined. In all the preceding formulæ, I have omitted the ambiguous sign $\pm$ when it occurs, thinking it perplexing and useless.

The first and second of (C) give

$$\sin\psi = \sqrt{\left(\frac{k^n}{h}\right)}\,.\,\sin\phi P\left(\frac{s^2\,.\,a\,.\,2r\omega - \sin^2\phi}{1 - k^2s^2\,.\,a\,.\,2r\omega\sin^2\phi}\right)$$
$$= \frac{\beta\tan\phi}{\sqrt{(1+\tan^2.\,\phi)}}\,.\,P\left(\frac{1 - \cot^2 a.2r\omega\tan^2\phi}{1 + \Delta^2 a.2r\omega\tan^2\phi}\right),$$

$$\cos\psi = \sqrt{\left(\frac{k''}{h}\right)} \cdot \cos\phi P\left(\frac{s^2 . a . (2r-1)\omega - \sin^2\phi}{1 - k^2 s^2 . a . 2r\omega \sin^2\phi}\right)$$

$$= \frac{1}{\sqrt{(1+\tan^2\phi)}} . P\left(\frac{1 - \cot^2 a\,(2r-1)\,\omega\tan^2\phi}{1 + \Delta^2 . a . 2r\omega\tan^2\phi}\right),$$

and therefore $\tan\psi = \beta\tan\phi . P\left(\dfrac{1 - \cot^2 a . 2r\omega\tan^2\phi}{1 - \cot^2 a(2r-1)\omega\tan^2\phi}\right)$.

Let R be the denominator of $\sin\psi$ and $\cos\psi$; then $\sin^2\psi + \cos^2\psi = 1$;

$R^2\sin^2\psi + R^2 . \cos^2\psi = R^2 = (1+\tan^2\phi) . P(1+\Delta^2 . a . 2r\omega\tan^2\phi)^2$.

Each factor of the first member of this last must be equal to the product of all those factors of the second member which divide it. Now $R\cos\psi + R\sin\psi\sqrt{(-1)} = 0$ gives $\tan\psi = \sqrt{(-1)}$, $R\cos\psi - R\sin\psi\sqrt{(-1)} = 0$ gives $\tan\psi = -\sqrt{(-1)}$; also $1 + \Delta . a . 2r\omega\tan\phi\sqrt{(-1)} = 0$ gives $\tan\phi = \dfrac{\sqrt{(-1)}}{\Delta . a . 2r\omega}$, which substituted in the expression of $\tan\psi$ gives it affirmative.

$$1 - \Delta . a . 2r\omega\tan\phi\sqrt{(-1)} = 0 \text{ gives } \tan\phi = \frac{-\sqrt{(-1)}}{\Delta . a . 2r\omega},$$

which gives $\tan\psi$ negative. The factors of the second member therefore which have their second term positive, divide that factor of the first whose second term is positive; and those which have the second term negative, divide that whose second term is negative. Consequently we have

$$R\cos\psi + R\sin\psi\sqrt{(-1)}$$
$$= \{1 + \tan\phi\sqrt{(-1)}\} . P\{1 + \Delta . a . 2r\omega\tan\phi\sqrt{(-1)}\}^2,$$

$$R\cos\psi - R\sin\psi\sqrt{(-1)}$$
$$= \{1 - \tan\phi\sqrt{(-1)}\} . P\{(1 - \Delta . a . 2r\omega\tan\phi\sqrt{(-1)}\}^2,$$

$$\frac{1 + \tan\psi\sqrt{(-1)}}{1 - \tan\psi\sqrt{(-1)}}$$
$$= \frac{1 + \tan\phi\sqrt{(-1)}}{1 - \tan\phi\sqrt{(-1)}} . P\left\{\frac{1 + \Delta . a . 2r\omega\tan\phi\sqrt{(-1)}}{1 - \Delta . a . 2r\omega\tan\phi\sqrt{(-1)}}\right\}^2.$$

Taking the logarithm of each member, we have by known formulæ,

$$\psi = \phi + 2\Sigma \operatorname{arc\,tan}\{\Delta . a . 2r\omega\tan\phi\} . r = 1, 2, \ldots \frac{n-1}{2} \quad . . (E).$$

Let us now differentiate (E), remembering that

$$d\psi = dv\sqrt{(1 - h^2\sin^2\psi)},\ d\phi = du\sqrt{(1 - k^2\sin^2\phi)};$$

and there results

$$dv\sqrt{(1 - h^2\sin^2\psi)} = du\sqrt{(1 - k^2\sin^2\phi)}$$
$$+ 2\,du\sqrt{(1 - k^2\sin^2\phi)} . \Sigma\,\frac{\Delta . a . 2r\omega}{1 - k^2 s^2 . a . 2r\omega\sin^2\phi}.$$

The last of (D) developed may be put under the following form:

$$\beta\sqrt{(1-h^2\sin^2\psi)} = \sqrt{(1-k^2\sin^2\phi)}$$
$$+ 2\sqrt{(1-k^2\sin^2\phi)}\,.\,\Sigma\,\frac{\Delta\,.\,a\,.\,2r\omega}{1-k^2s^2.a\,.\,2r\omega\sin^2\phi}\,;$$

the last but one divided by the last, member by member, gives $\dfrac{dv}{\beta} = du$, or $dv = \beta du$; which proves the success of the transformation.

The transformation we have obtained diminishes the modulus. We may easily derive one from it which increases it; thus make $\sin\psi = \tan\tau\sqrt{(-1)} = i\tan\tau$, $\sin\phi = i\tan\phi$; and (B) becomes

$$\frac{d\tau}{\sqrt{(1-h'^2\sin^2\tau)}} = \frac{\beta d\phi}{\sqrt{(1-k'^2\sin^2\phi)}}\,\ldots\ldots(F).$$

Developing (C), and putting $s.a.v = \sin\psi = i\tan\tau$, $s.a.u = \sin\phi = i\tan\phi$; those formulæ will give $\sin\tau$, $\cos\tau$, &c. expressed in terms of ϕ.

Suppose now n infinite, $k = 1$, $k' = 0$; then $K = \infty$, $K' = \dfrac{\pi}{2}$.

Also (F) gives $H' = \beta K' = \dfrac{\beta\pi}{2}$, or $\beta = \dfrac{2H'}{\pi}$. Again, (B) gives $H = \beta\omega$, or $\omega = \dfrac{H}{\beta} = \dfrac{\pi H}{2H'}$. When $k = 1$, $u = \displaystyle\int\frac{d\phi}{\cos\phi} =$ $\log\sqrt{\left(\dfrac{1+\sin\phi}{1-\sin\phi}\right)}$, which gives $\sin\phi = \dfrac{1-c^{-2u}}{1+c^{-2u}}$, c being the base of hyp. log. If for u in the last we put ω, 2ω, &c., or $\dfrac{\pi H}{2H'}$, $\dfrac{\pi H}{H'}$, &c. and for $\sin\phi$, $\sin am\,.\,\omega$, $\sin am\,.\,2\omega$, &c. we shall have these last quantities, and all other functions of $.am.\omega$, expressed in terms of $c^{-\frac{\pi H}{H}}$.

If we make $\tau = am\,.\,v'$, (F) becomes $v' = \beta\phi$; and $\sin\phi = \sin\dfrac{v'}{\beta} = \sin\dfrac{\pi v'}{2H'}$. Substituting these values in the formulæ obtained from (C) as above indicated, we shall have $\sin am\,.\,v'$, $\cos am\,.\,v'$ &c. expressed in terms of $\sin\dfrac{\pi v'}{2H'}$; or of the function itself. Thus the functions of the amplitude of v' may be expressed in terms of v' itself by infinite factorials or infinite series, as M. Jacobi has expressed them.

The transformation effected in this paper is that of M. Jacobi.

In page 47 of his work, changing his notation into mine where they differ, he has

$$s.a.v = \sqrt{\left(\frac{k^n}{h}\right)} s.a.u \; s.a\,(u + 4\omega) \ldots s.a\,\{u + 4\,(n - 1)\,\omega\}.$$

The middle factor is $s\,.\,a\,.\,\{u + 2\,(n - 1)\,\omega\}$; and the following factors easily reduce by (A) to

$$\pm\; s.a(u + 2\omega),\; \pm\; s.a(u + 6\omega), \text{ \&c.}$$

Also h' has the same value in this theory as λ' in Jacobi; see page 46. Therefore his transformation and mine are the same. I now proceed to point out an error into which he has fallen.

It has appeared that the factorial,

$$s.a.\omega\; s.a.3\omega\; s.a.5\omega \ldots s.a\,(2n - 1)\,\omega$$
$$= \{s.\omega\; a.s.a.3\omega \ldots s.a\,(n - 2)\,\omega\}^2.s.a.n\omega,$$

enters into the values of sin $am\,.\,v$, of h and h', and of β. If therefore $s.a.n\omega$ be nothing or infinite, it renders those values faulty. We have made $\omega = \frac{K}{n}$; but let $\omega = \frac{2K}{n}$; then sin $am\,.\,n\omega$ = sin $am\,.\,2K = 0$. In like manner, if m be any even integer, and $\omega = \frac{mK}{n}$, sin $am\,.\,n\omega$ = sin $am\,.\,mK = 0$.

We cannot therefore have $\omega = \frac{mK}{n}$, if m be an even integer.

Let $\sin\phi = \sqrt{(-1)}\,.\,\tan\psi = i\tan\psi$; see Jacobi, p. 34. Then

$$\frac{d\phi}{\sqrt{(1 - k^2\sin^2\phi)}} = \frac{id\psi}{\sqrt{(1 - k'^2\sin^2\psi)}}.$$

Whence the following,

$$\sin am\,(iu, k) = i\tan am\,(u, k');$$

$$\cos am\,(iu, k) = \frac{1}{\cos am\,(u, k')},\quad \Delta am\,(iu, k) = \frac{\Delta am\,(u, k')}{\cos am\,(u, k')} \ldots (G).$$

Now suppose $\omega = \frac{iK'}{n}$. This is making m in the last case equal to nothing, which is an even number. But sin $am\,.\,n\omega$ = sin $am\,.\,iK'$, or = sin $am\,(iK', k) = i\tan am\,(K', k') = i\,\infty$. This must render all the quantities above mentioned faulty, especially as all the other factors which enter into their values, will be finite. This is the value of ω in one of M. Jacobi's transformations, which therefore must be faulty.

Suppose in general with M. Jacobi, that $\omega = \frac{mK + m'iK'}{n}$, where m and m' are any integers affirmative or negative, but having no common divisor which also measures n. If m be

even, $\sin am \,.\, mK = 0$, $\cos am \,.\, mK = \pm 1$; and by the first of (A), $\sin am \,.\, n\omega = \sin am\,(mK + im'K') = \pm \sin am\,(im'K') = \pm \sin am\,(im'K', k) = \pm i \tan am\,(m'K', k')$ by $(G) = 0$ or α, according as m' is even or odd. This must render faulty all the quantities into the values of which $\sin am \,.\, n\omega$ enters; for the other factors are neither nothing nor infinite. M. Jacobi therefore has erred in supposing that m may be even.

VI.—ON THE SOLUTION OF FUNCTIONAL DIFFERENTIAL EQUATIONS.

By R. L. Ellis, B.A. Fellow of Trinity College.

It is well known that the solution of a considerable class of differential equations may be effected by means of differentiation. Clairaut's equation is a particular case of this class. We will begin by considering it.

$$y = px + fp \ldots . (1) \qquad \left(p = \frac{dy}{dx}\right),$$

where f denotes any given function.

Differentiating (1), we get

$$(x + f'p)\, q = 0 \ldots . (2),$$

hence $\qquad q = 0, \;$ or $\; x + f'p = 0 \ldots . (3).$

The first of these equations gives the complete integral. Being twice integrated it becomes $y = ax + b$; and on substitution in (1), we get $b = fa$, therefore $y = ax + fa \ldots\ldots (4)$ is the complete integral of (1).

It has always been supposed, in this and similar cases, that f must necessarily be a given function. But this condition is not essential: a differential equation, *e.g.* such as (1), will, when solved, give y as a function of x. Now the function f, which enters into (1) may, instead of being given, as is usually the case, be in some way dependent on the function which y is of x. Thus the form of f is unknown, until that of the latter function has been determined. It is evident that according to the classification proposed in the last number of the Journal, (1) is in all such cases a functional equation. For the unknown operation f is performed on p, which is itself the result of the unknown operation ψ' performed on x. (we suppose $y = \psi x$).

To differential functional equations, ordinary methods of solution do not, generally speaking, apply, because they require a knowledge of the forms of the functions on which they operate. But in the case before us, the differentiation and

subsequent substitution, by which (4) was derived from (1), are independent of any knowledge of the nature of f. Consequently (4) is always true.

Let us suppose, for instance, that $f = -m\psi$, m being a constant; then

$$\psi x - x\psi' x + m\psi\psi' x = 0 \ldots\ldots (5).$$

We are of course obliged to introduce a functional notation: (4) in this case becomes

$$\psi x = ax - m\psi a \ldots\ldots\ldots\ldots (6).$$

In order to determine ψa, put $x = a$;

$$\text{then} \quad \psi a = \frac{a^2}{1+m},$$

$$\text{and} \quad \psi x = ax - \frac{m}{1+m}a^2 \ldots\ldots (7),$$

which is a solution of (5).

In the ordinary cases of Clairaut's equation, the factor $x + f'p = 0$ leads to the singular solution; and so it does when f is an unknown function.

Thus, in the example just considered, as $f' = -m\psi'$, we shall have

$$m\psi'\psi' x = x \ldots\ldots (8).$$

Of this a solution is

$$\psi' x = \frac{x}{\sqrt{(m)}}.$$

Hence we get, by integration,

$$\psi x = \frac{x^2}{2\sqrt{(m)}} + C.$$

On substitution it is found that $C = 0$, therefore

$$\psi x = \frac{x^2}{2\sqrt{(m)}} \ldots\ldots (9)$$

is a new solution of (5), and perfectly distinct from (7).

If $m = 1$, (5) and (7) become respectively

$$\psi x - x\psi' x + \psi\psi' x = 0 \ldots\ldots (5'),$$

$$\psi x = ax - \tfrac{1}{2}a^2 \ldots\ldots\ldots\ldots (7');$$

in this case, (8) admits of a variety of simple solutions. Thus we shall have

$$\psi x = \tfrac{1}{4}c^2 - \tfrac{1}{2}(c-x)^2,$$

$$\psi x = \tfrac{1}{2} + \log x,$$

$$\&c. = \&c.$$

as singular solutions of (5').

The preceding remarks are sufficient to indicate the existence of a class of functional equations, to which a considerable portion of the theory of singular solutions may be applied. They appear therefore to possess some interest with reference to this theory, independently of the method they suggest for the solution of such equations.

In fact the theory can hardly be considered complete, unless some notice is taken of the equations of which we have been speaking. They have been excluded from it, because the function f, which they involve, is not, as in the ordinary case, a known function. But this, it has been already remarked, is not an essential distinction.

On the other hand, the method by which the singular solution is in the common theory deduced from the complete integral, does not apply to the cases now considered. It appears unnecessary to point out the reason of this difference.

With regard to the class of differential equations, which, like Clairaut's, separate into factors on differentiation, we may refer to Lagrange's *Leçons sur le Calcul des Fonctions*, l. 16^me^. He there shows that if a differential equation of the first order can be put into the form $M = fN$, where M and N are the values of a and b deduced from

$$F(xyab) = 0,$$

$$\frac{d}{dx} F(xyab) = 0;$$

then, when differentiated, it will resolve itself into two factors, one of which leads to the singular solution, and the other to the complete integral. (The latter is, as may readily be seen,

$$F(xyfb \,.\, b) = 0.)$$

The demonstration of this proposition is probably familiar to the majority of my readers, and I shall therefore not dwell upon it. Similar considerations apply to equations of higher orders.

Generalizing the remarks already made, we see that in the equation

$$M = fN,$$

the function f need not be a given one; it may be, in any way we please, dependent on the function which, in virtue of this equation, y is of x. In all such cases the equation in question is functional. Nevertheless, Lagrange's reasoning applies as much in these as in other cases. Let us take one or two examples of what has been said.

The following problem may be proposed.

Any point P of a certain curve is referred to the axis of x in M, and to that of y in N. MP is produced to Q; PQ is taken equal to a, and NQ touches the curve. Find its equation.

Let x, ψx be the co-ordinates of the point where NQ touches the curve.

$$ON = \psi x - x\psi' x. \quad NP = \frac{a}{\psi' x},$$

and as P is a point in the curve,

$$ON = \psi\,\{NP\}, \text{ or}$$

$$\psi x - x\psi' x = \psi\left(\frac{a}{\psi' x}\right) \ldots\ldots (10).$$

This is the equation of the problem. Differentiating it, we get

$$\psi'' x = 0,$$

$$x = \frac{a}{(\psi' x)^2}\,\psi'\left(\frac{a}{\psi' x}\right).$$

The former equation gives the complete integral, but, for a reason I shall hereafter notice, leads to no tangible solution of the problem; the latter corresponds to the singular solution.

In order to solve it, assume

$$\frac{a}{\psi' x} = \chi x;$$

$$\text{then} \quad \psi' \frac{a}{\psi' x} = \frac{c}{\chi^2 x} \text{ and } \frac{x}{\chi x} = \frac{\chi x}{\chi^2 x}.$$

Let $x = u_z$, $\chi x = u_{z+1}$, and therefore $\chi^2 x = u_{z+2}$.

Then $\dfrac{u_z}{u_{z+1}} = \dfrac{u_{z+1}}{u_{z+2}} = \dfrac{1}{C}$, where C is arbitrary;

$$\text{therefore } \chi x = Cx.$$

C is a function of z, which does not change when $z+1$ is substituted for z.

We confine ourselves to the only simple case, that in which it is an absolute constant; then

$$\psi' x = \frac{a}{Cx} = \frac{b}{x} \ldots. \; (bC = a)$$

and $\psi x = b \log \dfrac{x}{c}$ .... c being an arbitrary constant.

On substitution, we find $b = ae$; therefore

$$y = ae \log \frac{x}{c} \ldots. \; (11)$$

is a solution of the problem.

This is the equation of a logarithmic curve, which has therefore the required property. The method employed to resolve the equation in χx, namely

$$x\chi^2 x = (\chi x)^2,$$

is applicable to every equation of the form

$$F(x \,.\, \chi x \,\ldots\ldots\, \chi^n x) = 0 \ldots\ldots (12).$$

Every such equation may be at once reduced to the following equation in finite differences,

$$F(u_z \; u_{z+1} \ldots\ldots u_{z+n}) = 0 \ldots\ldots (13).$$

This reduction is in reality a particular case of an important transformation due to Mr. Babbage, which often enables us to solve functional equations of the higher orders.

In (12) we may write for χx, $\phi f \phi^{-1} x$.

Hence $\chi^2 x = \phi f^2 \phi^{-1} x$ &c. = &c., and (12) becomes

$$F(\phi x \,.\, \phi f x \ldots\ldots \phi f^n x) = 0 \ldots\ldots (14),$$

by putting ϕx for x; f being a known function, (14) is a functional equation of the first order.

Such is Mr. Babbage's method. Let $fx = 1 + x$; (14) becomes

$$F\{\phi x \,.\, \phi(1 + x) \ldots\ldots \phi(n + x)\} = 0,$$

and if we denote ϕx by u_x, and replace x by z, we shall obtain (13).

It must be admitted, that it is difficult to prove that the generality of 12) is not restricted by these transformations. They are however often useful, and serve to illustrate what was remarked in the last number, with respect to the affinity of functional equations, and equations in finite differences.

If, instead of (10), we had taken the more general equation

$$\psi x - x\psi' x = \psi\left(\frac{a}{\psi' x}\right) + A \ldots\ldots (15),$$

where A is an arbitrary constant, precisely the same method would have applied. In this case the factor $\psi'' x = 0$ would have led to the result

$$\psi x = ax + \beta,$$

and by substitution $\quad \beta = a + \beta + A.$

$$\text{therefore } \; a + A = 0, \quad \text{or } \beta = \infty.$$

Now in the case we have been considering, the former condition is not fulfilled; hence we must have $\beta = \infty$, and the geometrical interpretation of the complete integral is a right line at an infinite distance from the axis of abscissæ.

We not unfrequently meet with similar cases, in which the complete integral becomes nugatory or impossible in the process of introducing the necessary relation between its constants.

Under particular conditions, however, this difficulty does not occur, and then we obtain, what in the ordinary methods of discussing functional differential equations, appears to be a *conjugate* solution, unconnected with any other; (15) would be an instance of this, were $a + A = 0$.

I shall next consider a celebrated problem, first proposed by Euler, in the Petersburgh memoirs.

In a certain class of curves, the square of any normal exceeds the square of the ordinate drawn from its foot by a certain quantity a.

Let $y^2 = \psi x$ be the equation of the curve. The subnormal is therefore $\frac{1}{2}\psi' x$, and the equation of the problem consequently is

$$\psi(x + \tfrac{1}{2}\psi' x) = \psi x + \tfrac{1}{4}(\psi' x)^2 - a \quad \ldots . \text{ (16)}.$$

Differentiating this, we get

$$\psi'(x + \tfrac{1}{2}\psi' x) = \psi' x;$$

$$\text{or} \quad 1 + \tfrac{1}{2}\psi'' x = 0.$$

The first of these two equations leads to the singular solutions. In order to solve it, let

$$x + \tfrac{1}{2}\psi' x = \chi x,$$

$$\text{then} \quad \chi^2 x - 2\chi x + x = 0.$$

Hence by the transformation already noticed,

$$u_{z+2} - 2u_{z+1} + u_z = 0,$$

$$\text{whence} \quad u_z = Pz + zP_1 z,$$

where Pz and $P_1 z$ are functions of z, which remain unchanged when z increases by unity;

$$\text{therefore} \quad u_{z+1} = Pz + (z + 1) P_1 z.$$

Hence we have

$$\left.\begin{aligned} \tfrac{1}{2}\psi' x &= P_1 z \\ x &= Pz + zP_1 z \end{aligned}\right\} \text{ for the required solution.}$$

$$dx = (P'z + P_1 z + zP_1' z)\, dz;$$

$$\text{therefore} \quad y\, dy = P_1 z\,(P'z + P_1 z + zP_1' z)\, dz;$$

and integrating by parts, we get

$$\left.\begin{aligned} y^2 &= P_1 z\,(2Pz + zP_1 z) + \textstyle\int (P_1 z)^2\, dz \\ x &= Pz + zP_1 z \end{aligned}\right\} \ldots . \text{ (17)},$$

for a general solution of the proposed problem. (The parameter a is involved in $P_1 z$).

Let us suppose Pz and $P_1 z$ constant;

$$y^2 = a\,(2b + az) + a^2 z + C,$$

$$x = b + az;$$

$$\text{therefore} \quad y^2 = 2ax + C.$$

On substitution we find

$$a^2 = -a.$$

Thus, in order to a real result, we must suppose a negative, *e.g.* let $a = -k^2$; then

$$y^2 = 2kx + C \ldots\ldots (18),$$

the equation to a parabola, which accordingly is a solution of the problem, and the only simple one it admits of.

When $a = 0$, it becomes two straight lines parallel to the axis.

The other factor $1 + \frac{1}{2}\psi', x = 0$ gives, on integration,

$$\psi x + x^2 = ax + \beta \ldots\ldots (19),$$

the equation to a circle; but on substitution we find

$$\tfrac{1}{4}a^2 = \tfrac{1}{4}a^2 - a,$$

which leads to no result, unless $a = 0$.

A solution of this problem, by Poisson, is given at p. 591 of the last volume of Lacroix's great work. It is apparently equivalent in point of generality to (17); and the author points out its incompleteness in the case of $a = 0$. The preceding views show distinctly the nature and origin of the new solution which then presents itself. Mr. Babbage also has considered this problem at the end of his second essay on the Calculus of Functions (vide *Phil. Trans.* 1816, p. 253). But I believe it will be found that his solution is erroneous.

Notwithstanding the length this paper has already reached, I must endeavour to point out, as briefly as possible, my reasons for thinking so.

Mr. Babbage, confines himself to the case of $a = 0$. He begins by demonstrating the existence of a relation, equivalent, excepting a difference of notation, to $\psi'\{x + \frac{1}{2}\psi' x\} = \psi' x$, but in doing this, loses sight of the other factor $1 + \frac{1}{2}\psi' x = 0$.

This relation shows that $\psi' x$ is constant, for a series of points in the curve, and therefore, Mr. Babbage reasons, we may consider it as a constant in (16), which thus becomes an equation in finite differences. He integrates it on this supposition, and adds an arbitrary function of $\psi' x$, which has been treated as an absolute constant. The result is therefore

$$\psi x = \tfrac{1}{2}x\psi' x + f \cdot \tfrac{1}{2}\psi' x,$$

$$\text{or} \quad y^2 = xy\frac{dy}{dx} + f\left(y\frac{dy}{dx}\right) \ldots\ldots (20),$$

which is an ordinary differential equation.

This process appears to have been suggested by an incorrect analogy with the way in which arbitrary functions are introduced into partial differential equations.

A little consideration would have convinced Mr. Babbage, that by integrating (16) as an equation in finite differences, he only passed discontinuously from one ordinate of the curve to another, and therefore could not obtain a continuous relation between x and y. The legitimate result of his process is merely,

$$\psi\{x + \tfrac{1}{2}x\psi'x\} - \psi x = \frac{n}{4}(\psi'x)^2,$$

where n is any positive or negative integer. This is quite different from

$$\psi x = \tfrac{1}{2}x\psi'x + f\{\tfrac{1}{2}\psi'x\}.$$

In exemplifying equation (20), Mr. Babbage first supposes

$$f\left(y\,\frac{dy}{dx}\right) = \infty \times y\,\frac{dy}{dx},$$

and thus obtains the equation of a straight line parallel to Ox, as a solution of the problem, which undoubtedly it is.

In his next example $f\left(y\,\dfrac{dy}{dx}\right) = a^2$. By making the constant of integration imaginary, he gets $y^2 = a^2 - x^2$, the equation to a circle. But although this is also a real solution, it has no connection with the relation $\psi'\{x + \frac{1}{2}\psi'x\} = \psi'x$, from which it appears to be derived. It is, as we have seen, a particular case of the complete integral. Consequently if the method pursued had been correct, it could not have given this solution.

The preceding pages appear to contain the germ of a general theory of differential functional equations; a subject of great extent, and ultimately, perhaps, of considerable importance. But it cannot be denied, that hitherto the Calculus of Functions has not led to many results of much interest. Its value arises chiefly from the wide views it gives of the science of the combination of symbols.

VII.—ON CERTAIN DEFINITE INTEGRALS.

By Arthur Cayley, B.A. Trinity College.

In the first place, we shall consider the integral

$$V = \iint \ldots (\text{n times}) \frac{dx\,dy\,\ldots}{\{(a-x)^2 + (b-y)^2 \ldots\}^{\frac{n}{2}-1}},$$

the integration extending to all real values of the variables, subject to the condition

$$\frac{x^2}{h^2}+\frac{y^2}{h_1^{\,2}}+\ldots < \text{ or } = 1,$$

and the constants a, b, &c. satisfying the condition

$$\frac{a^2}{h^2}+\frac{b^2}{h_1^{\,2}}\ldots > 1.$$

We have

$$\frac{dV}{da}=-(\mathfrak{n}-2)\iint\ldots(\mathfrak{n}\text{ times}).\frac{(a-x)\,dx\,dy\ldots}{\{(a-x)^2+(b-y)^2\ldots\}^{\frac{1}{2}\mathfrak{n}}},$$

$$=-(\mathfrak{n}-2)\,\frac{2hh_{,}\ldots\pi^{\frac{1}{2}\mathfrak{n}}a}{\sqrt{(\xi+h^2)}\Gamma(\frac{1}{2}\mathfrak{n})}\int_0^1\frac{x^{\mathfrak{n}-1}\,.\,dx}{[\{\xi+h^2+(h_1^{\,2}-h^2)x^2\}\{\xi+h^2+(h_{,,}^{\,2}-h^2)x^2\}\ldots]^{\frac{1}{2}}},$$

ξ being determined by the equation

$$\frac{a^2}{\xi+h^2}+\frac{b^2}{\xi+h_{,}^{\,2}}\ldots=1,$$

by a formula in a paper, "On the Properties of a Certain Symbolical Expression," in the preceding number of this Journal. (ξ having been substituted for η^2.)

Let the variable x, on the second side of the equation, be replaced by ϕ, where

$$x^2=\frac{\xi+h^2}{\xi+h^2+\phi};$$

we have, without difficulty,

$$\frac{dV}{da}=-(\mathfrak{n}-2)\,.\,\frac{hh_{,}\ldots\pi^{\frac{1}{2}\mathfrak{n}}}{\Gamma(\frac{1}{2}\mathfrak{n})}\,.\,a\int_0^\infty\frac{d\phi}{(\xi+h^2+\phi)\sqrt{\Phi}},$$

where $\Phi=(\xi+h^2+\phi)(\xi+h_{,}^{\,2}+\phi)\ldots.$

and similarly,

$$\frac{dv}{db}=-(\mathfrak{n}-2)\,.\,\frac{hh_1\ldots\pi^{\frac{1}{2}\mathfrak{n}}}{\Gamma(\frac{1}{2}\mathfrak{n})}b\,.\int_0^\infty\frac{d\phi}{(\xi+h_{,}^{\,2}+\phi)\sqrt{\Phi}},$$

&c......

From these values it is easy to verify the equation

$$V=\frac{(\mathfrak{n}-2)\,hh_{,}\ldots\pi^{\frac{1}{2}\mathfrak{n}}}{2\,\Gamma(\frac{1}{2}\mathfrak{n})}\int_0^\infty\left(1-\frac{a^2}{\xi+h^2+\phi}-\frac{b^2}{\xi+h_{,}^{\,2}+\phi}\ldots\right)\cdot\frac{d\phi}{\sqrt{\Phi}}.$$

For this evidently verifies the above values of $\frac{dV}{da}$, $\frac{dV}{db}$, &c.

if only the term $\frac{dV}{d\xi}d\xi$ vanishes.

Now $$\frac{dV}{d\xi}=\frac{(\mathfrak{n}-2)\,hh_{,}\ldots\pi^{\frac{1}{2}\mathfrak{n}}}{2\Gamma(\frac{1}{2}\mathfrak{n})}\int_0^\infty d\phi\,.\,\frac{d}{d\xi}\,.\left(1-\frac{a^2}{\xi+h^2+\phi}\ldots\right)\cdot\frac{1}{\sqrt{\Phi}}.$$

Or, observing that

$$\frac{d}{d}\left(1-\frac{a^2}{\xi+h^2+\phi}-\ldots\right)\frac{1}{\sqrt{(\Phi)}}=\frac{d}{d\phi}\left(1\;\;\frac{a^2}{\xi+h^2+\phi}\;\ldots\right)\frac{1}{\sqrt{(\Phi)}},$$

and taking the integral from 0 to ∞,

$$\frac{dV}{d\xi}=-\frac{(n-2)hh_,\,..\pi^{\frac{1}{2}n}}{2\Gamma(\frac{1}{2}n)}\cdot\left(1-\frac{a^2}{\xi+h^2}-\frac{b^2}{\xi+h_,^2}\ldots\right)\frac{1}{\sqrt{\{(\xi+h^2)(\xi+h_,^2)\}}}=0,$$

in virtue of the equation which determines ξ.

No constant has been added to the value of V, since the two sides of the equation vanish as they should do, for $a, b\ldots$ infinite, for which values ξ is also infinite and the quantity

$$\left(1-\frac{a^2}{\xi+h^2+\phi}\;\ldots\right)\cdot\frac{1}{\sqrt{(\Phi)}},$$

which is always less than $\dfrac{1}{\sqrt{(\Phi)}}$, vanishes. Hence, restoring the values of V and Φ,

$$\iint\ldots(n \text{ times})\;\frac{dx\,dy\,..}{\{(a-x)^2+(b-y)^2\ldots\}^{\frac{1}{2}n-1}}$$

$$=\frac{(n-2)hh_,\ldots\pi^{\frac{1}{2}n}}{2\Gamma(\frac{1}{2}n)}\cdot\int_0^\infty\left(1-\frac{a^2}{\xi+h^2+\phi}-\frac{b^2}{\xi+h_,^2+\phi}\ldots\right)\cdot\frac{d\phi}{\sqrt{\{(\xi+h^2+\phi)(\xi+h_,^2+\phi)\}}}$$

.... the limits of the first side of the equation, and the condition to be satisfied by a, b, &c., also the equation for the determination of ξ, as above.

The integral

$$V'=\iint\,..(n \text{ times})\,.\,\frac{dx\,dy\,..}{\{(a-x)^2\ldots.\}^{\frac{1}{2}n}},$$

between the same limits, and with the same condition to be satisfied by the constants, has been obtained in the paper already quoted. Writing ξ instead of η^2, and $x^2=\dfrac{\xi}{\xi+\phi}$, we have

$$V'=\frac{hh_,\ldots\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)}\int_0^\infty\frac{d\phi}{\sqrt{\{(\xi+\phi)(\xi+h^2+\phi)(\xi+h_,^2+\phi)\}\,..}},$$

$$\text{where}\qquad\frac{a^2}{\xi+h^2}+\frac{b^2}{\xi+h_,^2}\ldots.=1.$$

Let $\nabla=\dfrac{d^2}{da^2}+\dfrac{d^2}{db^2}+\ldots$ Then by the assistance of a formula,

$$\nabla^q . \frac{1}{(a^2+b^2\ldots)^i}$$

$$= 2i.(2i+2)\ldots(2i+2q-2)(2i+2-\mathfrak{n})\ldots(2i+2q-\mathfrak{n}).\frac{1}{(a^2+b^2\ldots)^{i+q}}$$

given in the same paper, in which it is obvious that $a, b\ldots$ may be changed into $a-x, b-y$, &c. . . . : also putting $i = \frac{1}{2}\mathfrak{n}$, we have

$$\iint\ldots(\mathfrak{n}\text{ times}) . \frac{dx\,dy\ldots}{\{(a-x)^2+.\ \ldots\}^{\frac{1}{2}\mathfrak{n}+q}}$$

$$= \frac{hh_{,}\ldots\ldots\pi^{\frac{1}{2}\mathfrak{n}}}{2^{2q}.1.2\ldots q.\Gamma(\frac{1}{2}\mathfrak{n}+q)}\int_0^\infty d\phi . \nabla^q . \frac{1}{\sqrt{\{(\xi+\phi)(\xi+h^2+\phi)\ldots}}$$

Now in general

$$\nabla\chi\xi = \chi'\xi . \left(\frac{d^2\xi}{da^2}+\frac{d^2\xi}{db^2}\ldots.\right)+\chi''\xi . \left\{\left(\frac{d\xi}{da}\right)^2+\left(\frac{d\xi}{db}\right)^2\ldots.\right\}$$

$$= \chi'\xi . \Sigma\left(\frac{d^2\xi}{da^2}\right)+\chi''\xi . \Sigma\left(\frac{d\xi}{da}\right)^2, \text{ suppose.}$$

But $$\Sigma\frac{a^2}{(\xi+h^2)} = 1.$$

Hence $$\frac{2a}{\xi+h^2}-\left\{\Sigma\frac{a^2}{(\xi+h^2)^2}\right\} . \frac{d\xi}{da} = 0.$$

Whence $$\Sigma\left(\frac{d\xi}{da}\right)^2 = \frac{4}{\Sigma\frac{a^2}{(\xi+h^2)^2}}.$$

Also

$$\frac{2}{\xi+h^2}-4\frac{a}{(\xi+h^2)^2} . \frac{d\xi}{da}+2\left\{\Sigma\frac{a^2}{(\xi+h^2)^3}\right\}\left(\frac{d\xi}{da}\right)^2-\left\{\Sigma\frac{a^2}{(\xi+h^2)^2}\right\}\frac{d^2\xi}{da^2} = 0.$$

Whence taking the sum Σ, and observing that

$$-4\Sigma\frac{a}{(\xi+h^2)^2} . \frac{d\xi}{da} = -8 . \frac{\Sigma\frac{a^2}{(\xi+h^2)^3}}{\Sigma\frac{a^2}{(\xi+h^2)^2}} = -2\Sigma\frac{a^2}{(\xi+h^2)^3} . \Sigma\left(\frac{d\xi}{da}\right)^2,$$

$$2\Sigma\frac{1}{\xi+h^2}-\left\{\Sigma\frac{a^2}{(\xi+h^2)^2}\right\} . \Sigma\left(\frac{d^2\xi}{da^2}\right) = 0;$$

or $$\Sigma\left(\frac{d^2\xi}{da^2}\right) = \frac{2\Sigma\frac{1}{\xi+h^2}}{\Sigma\frac{a^2}{(\xi+h^2)^2}}.$$

$$\text{Or}\quad \nabla\chi\xi = \frac{2\chi'\xi\,.\,\Sigma\left(\dfrac{1}{\xi+h^2}\right) + 4\chi''\,.\,\xi}{\Sigma\,\dfrac{a^2}{(\xi+h^2)^2}}.$$

Hence the function

$$\int_0^\infty d\phi\,.\,\nabla\,.\,\frac{1}{\sqrt{\{(\xi+\phi)\,(\xi+h^2+\phi)\}\ldots\ldots}}$$

(observing that differentiation with respect to ξ is the same as differentiation with respect to ϕ) becomes integrable, and taking the integral between the proper limits, its value is

$$-\frac{2\chi_0\xi\,.\,\Sigma\,\dfrac{1}{\xi+h^2} + 4\chi_0'\,.\,\xi}{\Sigma\,\dfrac{a^2}{(\xi+h^2)^2}}.$$

$$\text{Where}\quad \chi_0\xi = \frac{1}{\sqrt{\{\xi\,.\,(\xi+h^2)\,(\xi+h_1^2)\}\ldots\ldots}}.$$

We have immediately

$$\frac{\chi_0'\,.\,\xi}{\chi_0\xi} = -\tfrac{1}{2}\,.\left(\frac{1}{\xi} + \Sigma\,\frac{1}{\xi+h^2}\right);$$

$$\text{or}\quad 2\chi_0\xi\,.\,\Sigma\left(\frac{1}{\xi+h^2}\right) + 4\chi_0'.\xi = -2\,\frac{\chi_0\xi}{\xi}.$$

Whence

$$\int_0^\infty d\phi.\nabla.\frac{1}{\sqrt{\{(\xi+\phi)(\xi+h^2+\phi)\}}} = \frac{2}{\xi\sqrt{\xi\,(\xi+h^2)}\left\{\dfrac{a^2}{(\xi+h^2)^2} + \dfrac{b^2}{(\xi+h_1^2)^2}\cdots\right\}}.$$

And hence restoring the value of ∇, and of the first side of the equation,

$$\iint\ldots\ldots(\mathfrak{n}\text{ times})\,.\,\frac{dxdy\ldots\ldots}{\{(a-x)^2+\ldots\ldots\}^{\frac{1}{2}\mathfrak{n}+q}}$$

$$= \frac{hh_1\ldots\ldots\pi^{\frac{1}{2}\mathfrak{n}}}{2^{2q-1}\,1\,.\,2\,..\,q\Gamma(\frac{1}{2}\mathfrak{n}+q)}\left(\frac{d^2}{da^2}+\frac{d^2}{db^2}\ldots\ldots\right)^{q-1}$$

$$\frac{1}{\xi\sqrt{\{\xi(\xi+h^2)\,(\xi+h_1^2)\ldots\ldots\}}\left\{\dfrac{a^2}{(\xi+h^2)^2}+\dfrac{b^2}{(\xi+h_1^2)^2}+\ldots\ldots\right\}},$$

with the condition $\dfrac{a^2}{\xi+h^2}+\dfrac{b^2}{\xi+h_1^2}\ldots\ldots=1$;

from which equation the differential coefficients of ξ, which enter into the preceding result, are to be determined.

In general if u be any function of $\xi, a, b\ldots\ldots$

$$\left(\frac{d^2}{da^2}+\frac{d^2}{db^2}\dots\right)u=\frac{4\,\dfrac{d^2u}{d\xi^2}+2\,\dfrac{du}{d\xi}\,.\,\Sigma\dfrac{1}{\xi+h^2}+4\Sigma\,\dfrac{a}{\xi+h^2}\cdot\dfrac{d^2u}{d\xi da}}{\Sigma\,\dfrac{a^2}{(\xi+h^2)^2}}+\Sigma\,\frac{d^2u}{da^2},$$

from which the values of the second side for $q = 1$, $q = 2$, &c. may be successively calculated.

The performance of the operation $\left(\frac{d}{da}\right)^p\left(\frac{d}{db}\right)^q\left(\frac{d}{dc}\right)^r$, upon the integral V', leads in like manner to a very great number of integrals, all of them expressible algebraically, for a single differentiation, renders the integration with respect to ϕ possible. But this is a subject which need not be further considered at present.

We shall consider, lastly, the definite integral

$$U=\iint\dots(\text{n times})\,.\,\frac{(a-x)f\left(\dfrac{x^2}{h^2}+\dfrac{y^2}{h_1^{\,2}}+\right)dxdy\dots}{\{(a-x)^2+(b-y)^2\dots\}^{\frac{1}{2}\text{n}}},$$

limits, &c. as before. This is readily deduced from the less general one

$$\iint\dots(\text{n times})\,.\,\frac{(a-x)\,dxdy\,\dots}{\{(a-x)^2+(b-y)^2\dots\}^{\frac{1}{2}\text{n}}}.$$

For representing this quantity by $F.(h, h_1\dots)$, it may be seen that

$$U=\int_0^1 f(m^2)\,.\,\frac{d}{dm}\,F.(mh, mh_1\dots)\,dm.$$

Now in the value of $F.(h, h_1 \dots)$, changing $h, h_1 \dots$ into $mh, mh_1 \dots$ also writing $m^2\phi$ instead of ϕ, and $m^2\xi'$ for ξ, we have

$$F.(mh, mh_1\dots)=\frac{hh_1\dots\pi^{\frac{1}{2}\text{n}}}{\Gamma(\frac{1}{2}\text{n})}\,a\,.\int_0^\infty\frac{d\phi}{(\xi'+h^2+\phi)\sqrt{(\Phi')}}.$$

Where $\qquad \Phi'=(\xi'+h^2+\phi)(\xi'+h_1^{\,2}+\phi)\dots$

And $\qquad \dfrac{a^2}{\xi'+h^2}+\dfrac{b^2}{\xi'+h_1^{\,2}}+\dots\dots=m^2.$

Hence $\qquad \dfrac{d}{dm}\,.\,F.(mh, mh_1\dots)=\dfrac{d\xi'}{dm}\,\dfrac{d}{d\xi'},\,F.(mh, mh_1\,..),$

$$=\frac{hh_1\dots\pi^{\frac{1}{2}\text{n}}}{\Gamma(\frac{1}{2}\text{n})}\,a\,.\,\frac{d\xi'}{dm}\,.\int_0^\infty d\phi\,.\,\frac{d}{d\xi'}\,\frac{1}{(\xi'+h^2+\phi)\sqrt{(\Phi')}},$$

or, observing that $\frac{d}{d\xi'}$ is equivalent to $\frac{d}{d\phi}$, and effecting the integrations between the proper limits,

$$\frac{d}{dm}\Gamma.(mh, mh_{,}...) = -\frac{hh_{,}\,...\,\pi^{\frac{1}{2}\mathfrak{n}}}{\Gamma(\frac{1}{2}\mathfrak{n})}a.\frac{1}{(\xi'+h^2)\sqrt{\{(\xi'+h^2)(\xi'+h_{,}^2)\}...}}.$$

Substituting this value, also $f\left\{\frac{a^2}{\xi'+h^2}+\frac{b^2}{\xi'+h_1^2}+..\right\}$ for $f(m^2)$, in the value of U, and observing that $m=0$ gives $\xi'=\infty$, $m=1$, $\xi'=\xi$, where ξ is a quantity determined as before by the equation

$$\frac{a^2}{\xi+h^2}+\frac{b^2}{\xi+h_{,}^2}+....=1,$$

we have

$$U=-\frac{hh_{,}...\pi^{\frac{1}{2}\mathfrak{n}}a}{\Gamma(\frac{1}{2}\mathfrak{n})}.\int_{\infty}^{\xi}\frac{f\left\{\frac{a^2}{\xi'+h^2}+....\right\}d\xi'}{(\xi'+h^2)\sqrt{\{(\xi'+h^2)(\xi'+h_{,}^2)...\}}},$$

or writing $\phi+\xi$ for ξ', $d\xi'=d\phi$, the limits of ϕ are $0, \infty$; or, inverting the limits and omitting the negative sign,

$$U=\frac{hh_{,}...\pi^{\frac{1}{2}\mathfrak{n}}.a}{\Gamma(\frac{1}{2}\mathfrak{n})}.\int_0^{\infty}\frac{f\left\{\frac{a^2}{\xi+h^2+\phi}+\frac{b^2}{\xi+h_{,}^2+\phi}+....\right\}d\phi}{(\xi+h^2+\phi)\sqrt{\{(\xi+h^2+\phi)(\xi+h_{,}^2+\phi)...\}}};$$

which, in the particular case of $\mathfrak{n}=3$, may easily be made to coincide with known results. The analogous integral

$$\iint...(\mathfrak{n}\text{ times}).\frac{f\left\{\frac{x^2}{h^2}+\frac{y^2}{h_{,}^2}+....\right\}dx\,dy....}{\{(a-x)^2+(b-y)^2....\}^{\frac{1}{2}\mathfrak{n}}}$$

is apparently not reducible to a single integral.

VIII.—DEMONSTRATIONS OF SOME GEOMETRICAL THEOREMS.

THE following geometrical theorems may perhaps be interesting to some of the readers of the *Mathematical Journal.* They are founded on the decomposition into quadratic factors of the trinomial $x^{2n}-2a^nx^n\cos\theta+a^{2n}$, and are therefore intimately related to the well-known theorem of Cotes. We assume then the theorem

$$x^{2n}-2a^nx^n\cos\theta+a^{2n}=(x^2-2ax\cos\frac{\theta}{n}+a^2)(x^2-2ax\cos\frac{\theta+2\pi}{n}+a^2)...$$

$$\{x^2-2ax\cos\frac{\theta+(n-1)\pi}{n}+a^2\}....\text{(A)}.$$

whence also, by making $x = a$, and writing $n\phi$ for $\dfrac{\theta}{2}$, we have the known theorem

$$\sin n\phi = 2^{n-1} \sin\phi \sin\left(\phi + \frac{\pi}{n}\right) \sin\left(\phi + \frac{2\pi}{n}\right) \ldots \sin\left(\phi + \frac{n-1}{n}\pi\right) \ldots \text{(B)}.$$

If a regular polygon of $2n$ sides circumscribe a circle, and if $p_1\, p_2 \cdots p_{2n-1}\, p_{2n}$ be the perpendiculars drawn on its sides from any point in the circumference of the circle, then

$$p_1\, p_2 \cdots p_{2n-1} + p_2\, p_n \cdots p_{2n} = \frac{r^n}{2^{r-2}},$$

where r is the radius of the circle.

Let the arc between the assumed point and the adjacent point of contact subtend an angle ϕ at the centre; then

$$p_1 = r\,(1 - \cos\phi) = 2r \sin^2 \frac{\phi}{2}$$

$$p_2 = r\left\{1 - \cos\left(\phi + \frac{2\pi}{2n}\right)\right\} = 2r \sin^2\left(\frac{\phi}{2} + \frac{\pi}{2n}\right),$$

$$p_3 = 2r \sin^2\left(\frac{\phi}{2} + \frac{\pi}{n}\right),$$

$$p_4 = 2r \sin^2\left(\frac{\phi}{2} + \frac{3\pi}{2n}\right) \text{ \&c. \&c.}$$

Hence $p_1\, p_3 \ldots p_{2n-1} = 2^n r^n \sin^2 \dfrac{\phi}{2} \sin^2\left(\dfrac{\phi}{2} + \dfrac{\pi}{n}\right) \ldots \sin^2\left(\dfrac{\phi}{2} + \dfrac{n-1}{n}\pi\right)$,

and by the theorem (B), writing $\frac{1}{2}\phi$ for ϕ we have

$$p_1\, p_3 \ldots p_{2n-1} = \frac{r^n}{2^{n-2}} \sin^2 \frac{n\phi}{2}.$$

In the same way we find

$$p_2\, p_n \cdots p_{2n} = \frac{r^n}{2^{n-2}} \sin^2 n\left(\frac{\phi}{2} + \frac{\pi}{2n}\right) = \frac{r^n}{2^{n-2}} \cos^2 n\frac{\phi}{2},$$

and therefore

$$p_1\, p_3 \cdots p_{2n-1} + p_2\, p_4 \cdots p_{2n} = \frac{r^n}{2^{n-2}} \ldots\ldots\ldots \text{(1)}.$$

Also, by multiplying the preceding expressions together,

$$p_1\, p_2 \cdots p_{2n-1}\, p_{2n} = \frac{r^{2n}}{2^{2n-2}} \sin^2 n\phi \ldots\ldots\ldots\ldots \text{(2)}.$$

If the given point be within the circumference of the circle, and if c be its distance from the centre, and ϕ the angle which c makes with the radius drawn to the adjacent point of contact, we have

$$p_1 = r - c\cos\phi = x^2 - 2xy\cos\phi + y^2,$$

if $\quad x^2 + y^2 = r,\;$ and $\;2xy = c.$

There are corresponding expressions for the other perpendiculars, and therefore

$$p_1\,p_3\cdots p_{2n-1} = (x^2 - 2xy\cos\phi + y^2)\ldots\left\{x^2 - 2xy\cos\left(\phi + \frac{n-1}{n}\pi\right) + y^2\right\}$$

$$= x^{2n} - 2x^n y^n \cos n\phi + y^{2n} \text{ by } (A).$$

Similarly

$$p_2\,p_n\,..p_{2n} = x^{2n} - 2x^n y^n \cos(n\phi + \pi) + y^{2n} = x^{2n} + 2x^n y^n \cos n\phi + y^{2n};$$

hence by subtraction,

$$p_2\,p_n\cdots p_{2n} - p_1 p_3 \ldots p_{2n-1} = 4x^n y^n \cos n\phi = \frac{c^n}{2^{n-2}}\cos n\phi \ldots\ldots (3).$$

If from the given point within the circumference, we draw perpendiculars $q_1\,q_2\cdots q_{2n}$ on the radii drawn to the points of contact, we have

$$q_1 = c\sin\phi,\quad q_2 = c\sin\left(\phi + \frac{\pi}{n}\right)\ldots q_{2n} = c\sin\left(\phi + \frac{2n-1}{n}\pi\right),$$

and therefore, considering magnitude only,

$$q_1\,q_2\cdots q_{2n} = \frac{c^{2n}}{2^{2n-2}}\sin^2 n\phi \ldots\ldots\ldots\ldots (4).$$

Mr. Leslie Ellis has shown (*Journal*, Vol. II. p. 272,) that, if $f(\phi)$ be a rational and integral function of $\sin\phi$ and $\cos\phi$, in which the highest power of these quantities is r,

$$f(\phi) + f\left\{\phi + \frac{2\pi}{n}\right\} .. + f\left\{\phi + \frac{n-1}{n}2\pi\right\} = \frac{n}{2\pi}\int_0^{2\pi} fx\,dx,$$

when $n > r$. By similar reasoning we may show that, when $n = r$,

$$f(\phi) + \ldots + f\left\{\phi + \frac{n-1}{n}2\pi\right\}$$

$$= \frac{n}{2\pi}\int_0^{2\pi} f(x)\,dx + \frac{n}{\pi}\int_0^{2\pi} dx\,f(x)\cos nx\,.\cos n\phi,$$

$$+ \frac{n}{\pi}\int_0^{2\pi} dx\,f(x)\sin nx\,.\sin n\phi.$$

Now if $p_1\,p_2\cdots p_n$ be the perpendiculars drawn from any point in the circumference of a circle, on a regular circumscribing polygon of n sides, we have

$$p_1^{\,n} = r^n(1 - \cos\phi)^n,\quad p_2^{\,n} = r^n\left\{1 - \cos\left(\phi + \frac{2\pi}{n}\right)\right\}^n \text{ \&c.}$$

Therefore

$$\frac{n}{2\pi}\int_0^{2\pi} dx\,f(x) = r^n\int_0^{2\pi} dx\,(1 - \cos x)^n = \frac{(2n-1)(2n-3)\ldots 3\,.\,1}{(n-1)(n-2)\ldots 2\,.\,1}r^n,$$

$$\frac{n}{\pi}\int_0^{2\pi} dx\,f(x)\cos nx = \frac{n}{\pi}r^n\int_0^{2\pi} dx\,(1 - \cos x)^n\cos nx = (-)^n\frac{n}{2^{n-1}}$$

$$\text{and}\quad \frac{n}{\pi}\int_0^{2\pi} dx\,(1-\cos nx)^n \sin nx = 0.$$

These results are easily arrived at, by observing that

$$(1-\cos x)^n = (-)^n\,\{(\cos x)^n - n(\cos x)^{n-1} + \&c.\},$$

$$= (-)^n\left\{\frac{\cos nx}{2^{n-2}} - \&c.\right\}$$

Hence,

$$\Sigma\,(p^n) = r^n\left\{\frac{(2n-1)(2n-3)\ldots.3.1}{(n-1)(n-2)\ldots.2.1} + (-)^n\frac{n}{2^{n-1}}\cos n\phi\right\};$$

but in the same manner as in (1), we find

$$p_1\,p_2\cdots p_n = \frac{r^n}{2^{n-2}}\sin^2 n\,\frac{\phi}{2} = \frac{r^n}{2^{n-1}}(1-\cos n\phi).$$

Therefore

$$\Sigma(p^n)+(-)^n p_1 p_2\cdots p_n = r^n\left\{\frac{(2n-1)(2n-3)\ldots3.1}{(n-1)(n-2)\ldots2.1} + (-)^n\frac{n}{2^{n-1}}\right\}\ldots(5).$$

In the same way, if c_1, c_2, &c. be the chords drawn from the given point to the angle of a regular inscribed polygon, we have

$$c_1^{\,2} = 2r^2\,(1-\cos\phi),\ \&c.$$

and therefore

$$\Sigma\,(c^{2n}) + (-)^n\,(c_1,\,c_2\,\ldots.\,c_n)^2$$

$$= r^n\left\{\frac{2^n\,(2n-1)\,(2n-3)\ldots.\ 3.1}{(n-1)\,(n-2)\ldots.\ 2.1} + (-)^n\,2n\right\}\ldots.\ (6).$$

Theorems of this kind are not confined to the circle: thus, in the parabola, if n tangents be drawn such that the arcs between the points of contact subtend equal angles at the focus, and if $p_1, p_2\ldots.p_n$, be the perpendiculars on them from the focus, we shall have

$$p_1.p_2\ldots.\ p_n = 2^{n-1}a^n\ \text{cosec}\ \frac{n\theta}{2},$$

a being one-fourth of the parameter, and θ being the angle which the axis of the parabola makes with the radius vector drawn to the adjacent point of contact. For the equation to the parabola being

$$\frac{1}{r} = \frac{1-\cos\theta}{2a} = \frac{1}{a}\sin^2\frac{\theta}{2},$$

$$\text{we have}\quad \frac{1}{p_1^{\,2}} = \frac{1}{ar} = \frac{1}{a^2}\sin^2\frac{\theta}{2},$$

$$\text{or}\quad \frac{1}{p_1} = \frac{1}{a}\sin\frac{\theta}{2},$$

and similarly for the others. Hence

$$\frac{1}{p_1 . p_2 \ldots . p_n} = \frac{1}{a^n} \sin\frac{\theta}{2} \sin\left\{\frac{\theta}{2} + \frac{\pi}{n}\right\} \ldots . \sin\left\{\frac{\theta}{2} + \frac{n-1}{n}\pi\right\}$$

$$= \frac{1}{2^{n-1}a^n} \sin\frac{n\theta}{2};$$

and therefore $p_1 . p_2 \ldots . p_n = 2^{n-1} a^n \operatorname{cosec} \frac{n\theta}{2} \ldots . (7).$

The theorems (1) and (5) were discovered by Professor Wallace, of Edinburgh, about the year 1791, but they have not, we believe, been as yet published.

D.

IX.—ON THE MOTION OF A SOLID BODY ABOUT ITS CENTRE OF GRAVITY.

I HAVE attempted in this paper the solution of the problem of the motion of a rigid body about its centre of gravity, fixed and acted on by no forces, by at once making use of the general dynamical principles, viz. those of the Conservation of *Vis Viva*, and of the Conservation of Areas, which furnish two of the integrals of the equation of motion. And though the final result thus obtained is not new, yet it is interesting to see the previous solution thus verified. I shall adopt the usual notation. The origin of the co-ordinates is the centre of gravity $\left.\begin{matrix} xyz \\ x_1\, y_1\, z_1 \end{matrix}\right\}$ the co-ordinates of an element δm of the body referred to fixed axes in space, and to the principal axes in the body respectively; ABC the moments of inertia about the principal axes $\omega_1\ \omega_2\ \omega_3$ the angular velocities $\left.\begin{matrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{matrix}\right\}$ the cosines of the inclinations of $x_1\ y_1\ z_1$ to $\left\{\begin{matrix} x \\ y \\ z \end{matrix}\right.$

The *vis viva* of the body (since the motion is wholly of rotation about the centre of gravity) is

$$\Sigma\delta m\ \{\omega'^2 (y^2 + z^2) + \omega''^2 (x^2 + z^2) + \omega'''^2 (x^2 + y^2)\},$$

$\omega'\ \omega''\ \omega'''$ having references to the fixed axes; and

$\left.\begin{matrix} y^2 + z^2 \\ z^2 + x^2 \\ x^2 + y^2 \end{matrix}\right\}$ being the squares of the distances of the elements δm from the axes of $\left\{\begin{matrix} x \\ y \\ z \end{matrix}\right.$.

The whole *vis viva* is therefore

$$\omega'^2\, \Sigma\delta m\,(y^2 + z^2) + \omega''^2\, \Sigma\delta m\,(z^2 + x^2) + \omega'''^2\, \Sigma\delta m\,(x^2 + y^2).$$

Now the whole *vis viva* being constant, it can make no difference about what axes we estimate it. Hence we have

$$A\omega_1^{\,2} + B\omega_2^{\,2} + C\omega_3^{\,2} = h \;\ldots\; (1).$$

Again, since there are no forces acting, the sum of the mass of each element into the projection of the area described in an unit of time on any one of the co-ordinate planes, is constant with reference to the time. Or, considering the plane yz,

$$\Sigma\delta m\left(z\,\frac{dy}{dt} - y\,\frac{dz}{dt}\right) = \text{constant}.$$

Now the projections on the planes z_1y_1, y_1x_1 x_1z_1, are

$$\left.\begin{array}{l} \Sigma\delta m\left(z_1\,\dfrac{dy_1}{dt} - y_1\,\dfrac{dz_1}{dt}\right) \\[2ex] \Sigma\delta m\left(y_1\,\dfrac{dx_1}{dt} - x_1\,\dfrac{dy_1}{dt}\right) \\[2ex] \Sigma\delta m\left(x_1\,\dfrac{dz_1}{dt} - z_1\,\dfrac{dx_1}{dt}\right) \end{array}\right\} \text{ or } \begin{array}{l} A\omega_1 \\[2ex] B\omega_2 \\[2ex] C\omega_3 \end{array}$$

Hence, by the usual method in Geometry, the projection on the plane yz will be the sum of these projections, each multiplied into the cosines of the angles between the planes; or we have

$$\begin{array}{r} \\ \text{similarly} \\ \\ \end{array} \left.\begin{array}{l} Aa_1\omega + Bb\omega_2 + Cc_3\omega = k_1 \\ Aa'\omega_1 + Bb'\omega_2 + Cc'\omega_3 = k_2 \\ Aa''\omega_1 + Bb''\omega_2 + Cc''\omega_3 = k_3 \end{array}\right\} (a).$$

Adding their squares, we have

$$A^2\omega_1^{\,2} + B^2\omega_2^{\,2} + C^2\omega_3^{\,2} = k^2 \;\ldots\; (2).$$

Again, from the first of equations (a),

$$A\left(a\,\frac{d\omega_1}{dt} + \omega_1\,\frac{da}{dt}\right) + B\left(b\,\frac{d\omega_2}{dt} + \omega_2\,\frac{db}{dt}\right) + C\left(c\,\frac{d\omega_3}{dt} + \omega_3\,\frac{dc}{dt}\right) = 0.$$

Now it may easily be proved (Poisson, Art. 411,) that

$$\left.\begin{array}{l} \dfrac{dc}{dt} = \omega_2 a - \omega_1 b \\[2ex] \dfrac{db}{dt} = \omega_1 c - \omega_3 a \\[2ex] \dfrac{da}{dt} = \omega_3 b - \omega_2 c \end{array}\right\}.$$

Substituting these in the above equation, it becomes

$$a\left\{A\frac{d\omega_1}{dt}+(C-B)\,\omega_2\,\omega_3\right\}+b\left\{B\frac{d\omega_2}{dt}+(A-C)\,\omega_1\omega_3\right\}$$
$$+c\left\{C\frac{d\omega_3}{dt}+(B-A)\,\omega_1\omega_2\right\}=0.$$

We shall obtain similar equations with $a'b'c'$ and $a''b''c''$ in place of abc. Multiplying these by $aa'a''$, and taking notice of the equations of condition,

$$ab+a'b'+a''b''=0,\quad ac+a'c'+a''c''=0,\quad \text{and } a^2+a'^2+a''^2=1,$$

$$\text{we have } A\frac{d\omega_1}{dt}+(C-B)\,\omega_2\omega_3=0\ldots.(3).$$

The equations (1), (2), (3), completely determine the motion, and eliminating $\omega_2\omega_3$, we have for finding w_1 in terms of t the equation

$$t=\int\frac{\pm A\sqrt{(BC)}\,d\omega_1}{\sqrt{\{A(A-C)\,\omega_1^2-(k^2-Ch)}\ \sqrt{(k^2-Bh-A(A-B)\omega_1^2}},$$

which may be reduced to an elliptic function; so that

$$t=\pm\sqrt{\left\{\frac{ABC}{(A-C)(k^2-Bh)}\right\}}\int\frac{d\phi}{\sqrt{(1-c^2\sin^2\phi)}},$$

$$\text{where } c^2=\frac{(Ah-k^2)}{(k^2-Bh)}\cdot\frac{C-B}{A-C},$$

$$\text{and } \omega_1^2=\frac{k^2-Ch}{A(A-C)}\cdot\frac{1}{1-c^2\sin^2\phi}.$$

Similarly ω_2, ω_3, may be found in terms of t, and thence the position of the body at any time.

α. β. γ.

X.—MATHEMATICAL NOTES.

1. *Demonstration of the principle of virtual velocities.*—If X, Y, Z, be the resolved parts of the moving forces which act on a particle m, the necessary and sufficient conditions of equilibrium are

$$X=0,\quad Y=0,\quad Z=0.$$

If a system of particles act on each other, the resolved parts of the force arising from the other particles which acts on each particle of the system, may be represented by the differential coefficients, taken with regard to the co-ordinates of the particle acted on, of a certain function; R of the mutual distances of the particles; and if X, Y, Z, represent the

resolved parts of all other forces acting on the particle, we have for each particle a set of equations of the form

$$X + \frac{dR}{dx} = 0, \quad Y + \frac{dR}{dy} = 0, \quad Z + \frac{dR}{dz} = 0.$$

If we multiply each of such equations by infinitesimal arbitrary quantities δx, δy, δz, and add them together, we obtain

$$\Sigma\,(X\delta x + Y\delta y + Z\delta z) + \delta R = 0.$$

If δx, δy, δz are proportional to any small possible displacements of the particle consistent with the preservation of the form of the system, $\delta R = 0$; and we obtain finally

$$\Sigma\,(X\delta x + Y\delta y + Z\delta z) = 0$$

as the equation which must be satisfied in all cases of equilibrium. This equation is the analytical expression of the principle of virtual velocities.

In order to deduce from this expression the six equations for the motion of a solid body, let $x + \delta x$, $y + \delta y$, $z + \delta z$, be the co-ordinates after displacement of a particle whose original co-ordinates are x, y, z; or, which is the same thing, be the co-ordinates of the particle referred to a system of axes which are slightly removed from the first, we have

$$x + \delta x = \alpha + ax + by + cz,$$
$$y + \delta y = \beta + a'x + b'y + c'z,$$
$$z + \delta z = \gamma + a''x + b''y + c''z,$$

where α, β, γ represent the co-ordinates of the displacement of the origin, a, b, c the cosines of the angles which the new axis of x makes with the original co-ordinate axes, and similarly for $a', b', c', a'', b'', c''$; so that we have the well known relations

$$aa' + bb' + cc' = 0$$
$$aa'' + bb'' + cc'' = 0$$
$$a'a'' + b'b'' + c'c'' = 0.$$

If the motions are infinitesimal, a, b', c'', differ from unity by infinitesimals of the second order, and the other quantities are infinitesimals of the first order; the equations are therefore reduced to

$$\delta x = \alpha + by + cz, \qquad a' + b = 0,$$
$$\delta y = \beta + a'x + c'z, \qquad a'' + c = 0,$$
$$\delta z = \gamma + a''x + b''y; \qquad b'' + c' = 0.$$

Eliminating a', b'', c'', we get

$$\delta x = a + by - a''z,$$
$$\delta y = \beta + c'z - bx,$$
$$\delta z = \gamma + a''x - a'y;$$

substituting these values in the equation

$$\Sigma\,(X\delta x + Y\delta y + Z\delta z) = 0, \text{ we obtain}$$

$$a\Sigma X + \beta\Sigma Y + \gamma\Sigma Z + b\Sigma\,(Xy - Yx) + c'\Sigma\,(Yz - Zy) + a''\Sigma\,(Zx - Xz) = 0,$$

and $a\beta\gamma$, $a''bc'$, being arbitrary, we obtain the common equations of equilibrium.

a. σ.

2. *Problem from the Papers of* 1842.—If $F(x, y, z) = \phi(u, v, w)$, where F is homogeneous of the n^{th} degree in x, y, z, and $u = \dfrac{dF}{dx}$, $v = \dfrac{dF}{dy}$, $w = \dfrac{dF}{dz}$; then

$$x = (n-1)\frac{d\phi}{du}, \quad y = (n-1)\frac{d\phi}{dv}, \quad z = (n-1)\frac{d\phi}{dw}.$$

Since F is homogeneous of n dimensions in x, y, z, we have

$$nF = x\frac{dF}{dx} + y\frac{dF}{dy} + z\frac{dF}{dz} = xu + yv + zw.$$

Hence $\quad ndF = xdu + ydv + zdw + udx + vdy + wdz,$

or $\quad (n-1)\,dF = xdu + ydv + zdw.$

But $(n-1)\,dF = (n-1)\,d\phi = (n-1)\left(\dfrac{d\phi}{du}du + \dfrac{d\phi}{dv}dv + \dfrac{d\phi}{dw}dw\right).$

Therefore equating the coefficients of the differentials,

$$x = (n-1)\frac{d\phi}{du}, \quad y = (n-1)\frac{d\phi}{dv}, \quad z = (n-1)\frac{d\phi}{dw}.$$

ε.

3. The following mathematical expression for the *discontinuous* law of the sliding scale in the new Corn Bill, may be interesting to some of our readers.

Let p be the price of the quarter of corn in shillings, d the duty; then the formula expressing d in terms of p is

$$d = \frac{20}{1 + 0^{50-p}} + \frac{1}{1 + 0^{p-50} + 0^{74-p}}\left(72 - p - \frac{2}{1 + 0^{53-p}} + \frac{2}{1 + 0^{p-67}}\right).$$

In like manner the expression for the proposed Income Tax on a property x, is

$$t = \frac{7}{240}\,\frac{1}{1 + 0^{x-150}}\left(x - \frac{300}{1 + 0^{150-x}}\right).$$

a. σ.